

Volichenko-type metasymmetry of braided Majorana qubits

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Abstract

This paper presents different mathematical structures connected with the parastatistics of braided Majorana qubits and clarifies their role; in particular, “mixed-bracket” Heisenberg-Lie algebras are introduced. These algebras belong to a more general framework than the *Volichenko algebras* defined in 1990 by Leites-Serganova as *metasymmetries* which do not respect even/odd gradings and lead to mixed brackets interpolating ordinary commutators and anticommutators.

In a previous paper braided $\mathbb{Z}_2$ -graded Majorana qubits were first-quantized within a graded Hopf algebra framework endowed with a braided tensor product. The resulting system admits truncations at roots of unity and realizes, for a given integer $s = 2, 3, 4, \dots$, an interpolation between ordinary Majorana fermions (recovered at $s = 2$) and bosons (recovered in the $s \rightarrow \infty$ limit); it implements a parastatistics where at most $s - 1$ indistinguishable particles are accommodated in a multi-particle sector.

The structures discussed in this work are:

- the quantum group interpretation of the roots of unity truncations recovered from a (superselected) set of reps of the quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$;
- the reconstruction, via suitable intertwining operators, of the braided tensor products as ordinary tensor products (in a minimal representation, the N -particle sector of the braided Majorana qubits is described by $2^N \times 2^N$ matrices);
- the introduction of mixed brackets for the braided creation/annihilation operators which define generalized Heisenberg-Lie algebras;
- the $s \rightarrow \infty$ untruncated limit of the mixed-bracket Heisenberg-Lie algebras producing parafermionic oscillators;
- (*meta*)symmetries of ordinary differential equations given by matrix Schrödinger equations in $0 + 1$ dimension induced by the braided creation/annihilation operators;
- in the special case of a third root of unity truncation, a nonminimal realization of the intertwining operators defines the system as a ternary algebra.

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1 Introduction

Emergent Majorana fermions ($\mathbb{Z}_2$ -graded Majorana qubits) have been intensively investigated in the light of the Kitaev's proposal [1] (see also [2–4]) to use them for encoding topological quantum computations which offer topological protection from quantum decoherence.

It was shown in [5] that the braiding of $\mathbb{Z}_2$ -graded Majorana qubits can be realized within a graded Hopf algebra endowed with a braided tensor product (this approach was based on the framework proposed in [6]). In the present paper the First Quantization of $\mathbb{Z}_2$ -graded braided Majorana qubits is shown to be connected and expressed by several different constructions. We briefly sketch them.

At first this paper solves a question left unanswered in [5, 7]: how to directly relate the observed roots-of-unity truncations of the multi-particle braided spectra to quantum group reps data; it is shown that the truncations are also recovered from (superselected) representations of the quantum group $\mathcal{U}_q(\mathfrak{osp}(1|2))$.

On the technical side, the realization of the braided tensor products in terms of intertwining operators acting on ordinary tensor products paves the way for the further constructions presented in the paper.

The next relevant topic regards the parastatistics induced by the braided Majorana qubits. An s^{th} - root of unity truncation realizes, for a given integer $s = 2, 3, 4, \dots$, an interpolation between ordinary Majorana fermions (recovered at $s = 2$) and the untruncated case recovered in the $s \rightarrow \infty$ limit. For the s -th root at most $s - 1$ indistinguishable particles are accommodated in a multi-particle sector.

These “level- s parastatistics” are reproduced via the introduction of generalized Heisenberg-Lie algebras defined by “mixed brackets” which interpolate ordinary commutators (denoted as “[$\cdot, \cdot$]”) and anticommutators (denoted as “[$\cdot, \cdot$]”). Given two operators X, Y the $(\cdot, \cdot)_{\vartheta_{XY}}$ mixed bracket is defined as

$$(X, Y)_{\vartheta_{XY}} := i \sin(\vartheta_{XY}) \cdot [X, Y] + \cos(\vartheta_{XY}) \cdot \{X, Y\},$$

where the ϑ_{XY} angle is consistently determined.

These new mixed-bracket Heisenberg-Lie algebras broadly fit into the notion of “symmetries wider than supersymmetry” presented by Leites and Serganova in [8] (see also [9]); this notion concerns the existence of statistics-changing maps which do not preserve the $\mathbb{Z}_2$ -grading of ordinary Lie superalgebras.

The Leites-Serganova construction is focused on “Volichenko algebras” which act as *meta-symmetries* which do not respect even/odd gradings. A Volichenko algebra satisfies a “metaabelianess condition” (*metaabelianess* means that, for any X, Y, Z triple of operators, the ordinary $[[X, Y], Z] = 0$ commutators are vanishing). The introduction of mixed brackets which interpolate ordinary commutators/anticommutators is also a natural ingredient of the Leites-Serganova construction, see e.g. formula (1.3.2) of reference [9].

The mixed-bracket Heisenberg-Lie algebras are not Volichenko algebras since they violate the metaabelianess condition for some choices of the X, Y, Z operators. Nevertheless, they satisfy an alternative identity involving the mixed brackets: for any X, Y, Z triple of operators, the relation $(X, (Y, Z)) = 0$ is satisfied. This identity can be referred to as “metaabelianess with respect to the mixed brackets”.

Furthermore, the notion of *metasymmetry* can be applied to the mixed-bracket Heisenberg-Lie algebras. Indeed, it is shown in Section 10 that these mixed-bracket algebras close dynamical (meta)symmetries for given systems of Matrix Ordinary Differential Equations.

It is pointed out that in the $s \rightarrow \infty$ untruncated limit the mixed-bracket Heisenberg-Lie algebras reduce to (anti)commutators defining a set of parafermionic oscillators.

In Appendix **A** we clarify that the mixed-bracket parastatistics induced by the generalized Heisenberg-Lie algebras *are not* reproduced by another type of mixed-bracket construction, the quonic oscillator interpolation introduced by Greenberg [10] and by Mohapatra [11].

In Appendix **B** a nonminimal realization of the intertwining operators for the third root of unity allows to make contact with the ternary algebras entering the applications of ternary physics (for ternary algebras see [12] and references therein).

We postpone to the Conclusions a more detailed assessment of the found connections relating multi-particle braided $\mathbb{Z}_2$ -graded Majorana qubits and the various afore-mentioned mathematical structures (quantum group reps at roots of unity, truncated parastatistics and the untruncated $s \rightarrow \infty$ limit, metasymmetries induced by mixed-bracket algebras, etc.).

The scheme of the paper is the following:

- In Section **2** we revisit the [5] construction of the multi-particle $\mathbb{Z}_2$ -graded braided Majorana qubits obtained from a graded Hopf algebra endowed with a compatible braided tensor product;
- In Section **3** we present different suitable parametrizations of the truncations of the spectra at roots of unity;
- In Section **4** we recover the roots-of-unity truncations from a quantum group perspective, pointing out that it is given by (superselected) reps of the quantum supergroup $\mathcal{U}_q(\mathfrak{osp}(1|2))$;
- In Section **5** the braiding of Majorana qubits is realized via intertwining operators acting on ordinary (i.e., not braided) tensor products;
- In Section **6** the indistinguishability of identical particles is recovered as a superselection;
- Section **7** presents, following Leites-Serganova's papers, a sketchy introduction to the notion of the Volichenko algebras and their related metasymmetries;
- Section **8** presents the generalized "mixed-bracket" Heisenberg-Lie algebras which interpolate bosons/fermions and reproduce the multi-particle sectors of the graded Majorana qubits;
- In Section **9** it is shown that the $s \rightarrow \infty$ limit of the mixed-bracket Heisenberg-Lie algebras reduce to (anti)commutators describing a set of parafermionic oscillators;
- Section **10** presents the mixed-bracket Heisenberg-Lie algebras as dynamical metasymmetries of an ordinary differential Matrix Schrödinger equation in $0 + 1$ dimensions;
- In the Conclusions we give comments about the link of the presented constructions with parastatistics and discuss future perspectives;
- Appendix **A** clarifies the difference between generalized mixed-bracket Heisenberg-Lie algebras and the parastatistics induced by quons;
- In Appendix **B** a nonminimal realization of the intertwining operators for the third root of unity is related to ternary algebras.

2 The braided Majorana qubits revisited

It was shown in [5] that $\mathbb{Z}_2$ -graded Majorana qubits can be braided within a First Quantization formalism. We revisit in this Section the main ingredients of this construction which makes use of a graded Hopf algebra endowed with a braided tensor product.

The starting point of [5] is the $\mathfrak{gl}(1|1)$ superalgebra. A single Majorana fermion describes a

$\mathbb{Z}_2$ -graded qubit which defines a bosonic vacuum state $|0\rangle$ and a fermionic excited state $|1\rangle$:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1)$$

The following operators, acting on the $\mathbb{Z}_2$ -graded qubit, close $\mathfrak{gl}(1|1)$:

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \delta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (2)$$

The (anti)commutators are

$$\begin{aligned} [\alpha, \beta] &= \beta, & [\alpha, \gamma] &= -\gamma, & [\alpha, \delta] &= 0, & [\delta, \beta] &= -\beta, & [\delta, \gamma] &= \gamma, \\ \{\beta, \beta\} &= \{\gamma, \gamma\} = 0, & \{\beta, \gamma\} &= \alpha + \delta. \end{aligned} \quad (3)$$

The diagonal operators α, δ are even, while β, γ are odd (γ is the fermionic creation operator).

The excited state $|1\rangle$ is a Majorana fermion which coincides with its own antiparticle. The diagonal operator δ can be assumed to be the Hamiltonian H_1 (therefore, $H_1 \equiv \delta$).

The Universal Enveloping Algebra $\mathcal{U} \equiv \mathcal{U}(\mathfrak{gl}(1|1))$, which is a graded Hopf algebra, allows to introduce the multi-particle $\mathbb{Z}_2$ -graded qubits. A non-trivial braiding of the Majorana qubits is realized by introducing, following the [6] prescription, a braided tensor product (here denoted as “ $\otimes_{br}$ ”) which is compatible with the $\mathfrak{gl}(1|1)$ superalgebra. This means in particular that, for a nonvanishing complex parameter $t \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the creation operator γ can be assumed to satisfy the relation

$$(\mathbb{I}_2 \otimes_{br} \gamma) \cdot (\gamma \otimes_{br} \mathbb{I}_2) = B_t \cdot (\gamma \otimes_{br} \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes_{br} \gamma) \equiv B_t \cdot (\gamma \otimes_{br} \gamma), \quad (4)$$

where the “ $\cdot$ ” symbol denotes the standard matrix multiplication; here and in the following “ $\mathbb{I}_n$ ” denotes the $n \times n$ identity matrix. The t -dependent 4×4 constant matrix B_t is invertible for $t \neq 0$. It can be assumed to be the R -matrix of the Alexander-Conway polynomial in the linear crystal rep on exterior algebra [13]; it is related, see [14], to the Burau representation of the braid group. The matrix B_t is given by

$$B_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-t & t & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -t \end{pmatrix}. \quad (5)$$

The $\otimes_{br}$ braided tensor product defined in (4) satisfies the required [6] compatibility conditions; this is a consequence of the B_t matrix satisfying the braid relation

$$(B_t \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes B_t) \cdot (B_t \otimes \mathbb{I}_2) = (\mathbb{I}_2 \otimes B_t) \cdot (B_t \otimes \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes B_t). \quad (6)$$

2.1 The construction of the multi-particle states

Let $\mathcal{H}$ be the single-particle Hilbert space spanned by $|0\rangle, |1\rangle$ entering (1). The N -particle Hilbert space $\mathcal{H}_N$ is a subset of N tensor products of the single-particle Hilbert spaces:

$$\mathcal{H}_N \subset \mathcal{H}^{\otimes N}. \quad (7)$$

The N -particle vacuum state $|0\rangle_N$ is defined as

$$|0\rangle_N = |0\rangle \otimes \dots \otimes |0\rangle \quad (N \text{ times}), \quad (8)$$

while the N -particle excited states are obtained by repeatedly applying on $|0\rangle_N$ the coproducts of the creation operator γ .

The coproduct map Δ sends

$$\Delta : \mathcal{U} \rightarrow \mathcal{U} \otimes_{br} \mathcal{U}, \quad \Delta^{(N+1)} := (\Delta \otimes_{br} id) \Delta^{(N)} = (id \otimes_{br} \Delta) \Delta^{(N)} \in \mathcal{U}^{\otimes_{br} N}. \quad (9)$$

The second relation expresses the coassociativity of the coproduct.

The further property

$$\Delta(U_A U_B) = \Delta(U_A) \Delta(U_B) \quad \text{for any } U_A, U_B \in \mathcal{U} \quad (10)$$

implies that the action on any given $U \in \mathcal{U}(\mathfrak{gl}(1|1))$ is recovered from the action of the coproduct on the Hopf algebra unit $\mathbf{1}$ and on the primitive elements $\zeta \in \mathfrak{gl}(1|1)$; they are respectively given by

$$\Delta(\mathbf{1}) = \mathbf{1} \otimes_{br} \mathbf{1}, \quad \Delta(\zeta) = \mathbf{1} \otimes_{br} \zeta + \zeta \otimes_{br} \mathbf{1}. \quad (11)$$

In our conventions a hat denotes the evaluation of the coproduct on a given representation R of the Universal Enveloping Algebra $\mathcal{U}$, so that

$$\text{for } R : \mathcal{U} \rightarrow V, \quad \widehat{\Delta} := \Delta|_R \in \text{End}(V \otimes V), \quad \text{with } \widehat{\Delta}(U) \in V \otimes V \quad (12)$$

and, similarly, $\widehat{\Delta^{(N)}}(U) \in V \otimes \dots \otimes V$ taken $N+1$ times. For the case under consideration the vector space V is given by $\mathcal{H}$. It deserves being pointed out that the presence of a hat indicates an ordinary $m \times m$ matrix; this is why ordinary tensor products appear in formula (12), while braided tensor products appear in (9).

The N -particle Hilbert space $\mathcal{H}_N$ is spanned by the normalized vectors $|n\rangle_{t,N}$, where the integer n labels the n -th excited state; we have

$$|n\rangle_{t,N} \propto (\widehat{\Delta^{(N-1)}}(\gamma))^n |0\rangle_N, \quad \text{for } n = 0, 1, 2, \dots \quad (13)$$

The N -particle Hamiltonian H_N is expressed in terms of $H_1 = \delta = \text{diag}(0, 1)$ as

$$H_N := \widehat{\Delta^{(N-1)}}(H_1). \quad (14)$$

At the lowest orders the $N = 2, 3$ -particle Hamiltonians are

$$\begin{aligned} H_2 &= H_1 \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes H_1 = \text{diag}(0, 1, 1, 2), \\ H_3 &= H_1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes H_1 \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes H_1 = \text{diag}(0, 1, 1, 2, 1, 2, 2, 3). \end{aligned} \quad (15)$$

The $N = 2, 3$ -particle coproducts of the creation operator γ are

$$\begin{aligned} \Delta(\gamma) &= \gamma \otimes_{br} \mathbb{I}_2 + \mathbb{I}_2 \otimes_{br} \gamma, \\ \Delta^{(2)}(\gamma) &= \gamma \otimes_{br} \mathbb{I}_2 \otimes_{br} \mathbb{I}_2 + \mathbb{I}_2 \otimes_{br} \gamma \otimes_{br} \mathbb{I}_2 + \mathbb{I}_2 \otimes_{br} \mathbb{I}_2 \otimes_{br} \gamma. \end{aligned} \quad (16)$$

For later convenience it is useful to split the $N = 2, 3$ -particle coproducts of the creation operator into the following building blocks:

$$\begin{aligned} \text{For } N = 2 : \quad A_1^\dagger &:= \gamma \otimes_{br} \mathbb{I}_2, \quad A_2^\dagger := \mathbb{I}_2 \otimes_{br} \gamma. \\ \text{For } N = 3 : \quad B_1^\dagger &:= \gamma \otimes_{br} \mathbb{I}_2 \otimes_{br} \mathbb{I}_2, \quad B_2^\dagger := \mathbb{I}_2 \otimes_{br} \gamma \otimes_{br} \mathbb{I}_2, \quad B_3^\dagger := \mathbb{I}_2 \otimes_{br} \mathbb{I}_2 \otimes_{br} \gamma. \end{aligned} \quad (17)$$

Equation (4) implies in particular that

$$\begin{aligned} A_2^\dagger \cdot A_1^\dagger &= -t A_1^\dagger \cdot A_2^\dagger \quad \text{and} \\ B_2^\dagger \cdot B_1^\dagger &= -t B_1^\dagger \cdot B_2^\dagger, \quad B_3^\dagger \cdot B_1^\dagger = -t B_1^\dagger \cdot B_3^\dagger, \quad B_3^\dagger \cdot B_2^\dagger = -t B_2^\dagger \cdot B_3^\dagger. \end{aligned} \quad (18)$$

By taking into account the nilpotency of γ we get

$$\gamma^2 = 0 \Rightarrow (A_i^\dagger)^2 = 0 \quad \text{for } i = 1, 2 \quad \text{and} \quad (B_j^\dagger)^2 = 0 \quad \text{for } j = 1, 2, 3. \quad (19)$$

It follows that the coproduct of the 2-particle second-excited state is given by

$$(\Delta(\gamma))^2 = (1-t)A_1^\dagger \cdot A_2^\dagger. \quad (20)$$

The splitting into building blocks is extended to the coproducts of the N -particle sectors. We can set, for $k = 1, \dots, N$,

$$\begin{aligned} A_{k;N}^\dagger &:= \underbrace{\mathbb{I}_2 \otimes_{br} \dots \otimes_{br} \mathbb{I}_2}_{k-1 \text{ times}} \otimes_{br} \gamma \otimes_{br} \underbrace{\mathbb{I}_2 \otimes_{br} \dots \otimes_{br} \mathbb{I}_2}_{N-k \text{ times}}, \\ \Delta^{(N-1)}(\gamma) &= \sum_{k=1}^{N-1} A_{k;N}^\dagger. \end{aligned} \quad (21)$$

Formulas (17,18) are recovered from the positions $A_i^\dagger \equiv A_{i;2}^\dagger$ and $B_j^\dagger \equiv A_{j;3}^\dagger$.

In the next Section we summarize the results concerning the multi-particle spectra of the braided Majorana qubits and the truncations recovered at certain t roots of unity.

3 The truncations at roots of unity

In this Section we summarize the results concerning the multi-particle energy spectra of the braided $\mathbb{Z}_2$ -graded Majorana qubits. Since the derivation of these results has been presented in [5], it will not be repeated here. We stress in particular the role of the truncations which are recovered at certain roots of unity of the braiding parameter t entering (4,5). Different useful parametrizations of t , which will be used in the following, are introduced.

The $k = 1, \dots, N$ building blocks $A_{k;N}^\dagger$ entering (21) satisfy the braiding relations

$$A_{k';N}^\dagger \cdot A_{k;N}^\dagger = -t A_{k;N}^\dagger \cdot A_{k';N}^\dagger \quad \text{for } k < k'. \quad (22)$$

In consequence of that the operators $(\widehat{\Delta^{(N-1)}(\gamma)})^n$ (entering the (13) vectors $|n\rangle_{t,N}$ spanning the N -particle Hilbert space $\mathcal{H}_N$) are proportional to a factor $f_n(t)$, expressed by the t -dependent polynomials $b_k(t)$:

$$f_n(t) = \prod_{k=1}^n b_k(t), \quad \text{where} \quad b_k(t) = \sum_{j=0}^{k-1} (-t)^j. \quad (23)$$

A nonvanishing vector $|n\rangle_{t,N}$ is an eigenvector of the N -particle Hamiltonian H_N with energy eigenvalue $E_{n;t,N} = n$:

$$H_N |n\rangle_{t,N} = E_{n;t,N} |n\rangle_{t,N}, \quad \text{where} \quad E_{n;t,N} = n. \quad (24)$$

Therefore, the zeros of the $f_n(t)$ polynomial factors (which are induced by the zeros of the $b_k(t)$ polynomials defined in (23)), produce truncations at the corresponding energy level spectra.

It is easily shown that the zeros of the $b_k(t)$ polynomials lie in the $|t| = 1$ circle and are given by *certain* roots of unity. The *untruncated* spectrum is recovered for all other values of t , both

the nonvanishing complex parameter t satisfying $|t| \neq 1$ and the generic values of t belonging to the $|t| = 1$ circle which are not roots of the $b_k(t)$ polynomials. Without loss of generality, the physics of the multi-particle braided Majorana qubits (both truncated and untruncated sectors) can be derived by restricting t to belong to the $|t| = 1$ circle.

A first parametrization of the unit circle, see [7], is introduced by setting

$$t = e^{i\pi f}, \quad \text{for real values } f \in [0, 2[. \quad (25)$$

A relevant notion is that of “level- s ” root of unity for $s = 2, 3, 4, \dots$. It is defined as follows. The $b_k(t)$ polynomial is of order $k - 1$ (we have $b_1(t) = 1$, $b_2(t) = 1 - t$, $b_3(t) = 1 - t + t^2$, $\dots$). Its $k - 1$ roots are roots of unity. A level- s root of unity t_s corresponds to a solution of the equation

$$b_s(t_s) = 0 \quad (26)$$

which *is not* a root of any previous $b_k(t)$ polynomial for $k < s$; namely

$$b_k(t_s) \neq 0 \quad \text{for } k = 2, 3, \dots, s - 1. \quad (27)$$

Denoted as L_s the set of level- s roots of unity $t_s^{(j)}$ we have, at the lowest orders, the following representatives:

$$\begin{aligned} L_2 &: t_2^{(1)} = 1, & \text{implied by } b_2(t) = 1 - t = 0, \\ L_3 &: t_3^{(1)} = e^{\frac{i\pi}{3}}, \quad t_3^{(2)} = e^{\frac{5i\pi}{3}}, & \text{implied by } b_3(t) = 1 - t + t^2 = 0, \\ L_4 &: t_4^{(1)} = e^{\frac{i\pi}{2}}, \quad t_4^{(2)} = e^{\frac{3i\pi}{2}}, & \text{implied by } b_4(t) = 1 - t + t^2 - t^3 = 0. \end{aligned} \quad (28)$$

One should note that $t = 1$, which is the third root of $b_4(t) = 0$, is not a level-4 root of unity since it is already a solution of $b_2(t) = 0$.

The $t = -1$ root of unity *is not a root* of any $b_n(t)$ polynomial since, for $n = 1, 2, 3, \dots$,

$$b_n(t)|_{t=-1} = n \neq 0. \quad (29)$$

With a slight abuse of language we can say that $t = -1$ is a level- ∞ root, denoted as L_∞ .

In terms of f introduced in (25), the following fractions correspond to the first cases of level- s roots:

$$L_\infty : f_\infty = 1; \quad L_2 : f_2 = 0; \quad L_3 : f_3 = \frac{1}{3}, \frac{5}{3}; \quad L_4 : f_4 = \frac{1}{2}, \frac{3}{2}; \quad L_5 : f_5 = \frac{1}{5}, \frac{3}{5}, \frac{7}{5}, \frac{9}{5}. \quad (30)$$

The physical significance of a level- s root of unity is that the corresponding braided multi-particle Hilbert space can accommodate at most $s - 1$ braided Majorana particles.

The special point $t = 1$, which is the level-2 root of unity, gives the ordinary total antisymmetrization of the fermionic wavefunctions. The level-2 root of unity $t = 1$ encodes the Pauli exclusion principle of ordinary fermions.

One should note that the physics does not depend on the specific value of t , but only on its root of unity level. A generic t which does not coincide with a root of unity produces the same untruncated spectrum as the L_∞ root of unity $t = -1$.

The following N -particle energy spectra are derived. The energy levels are given by integer numbers and are not degenerate. The two big classes are:

Class a, truncated L_s level: the N -particle energy eigenvalues E are

$$\begin{aligned} E &= 0, 1, \dots, N & \text{for } N < s, \\ E &= 0, 1, \dots, s-1 & \text{for } N \geq s; \end{aligned} \quad (31)$$

in this case a plateau is reached at the maximal energy level $s-1$; this is the maximal number of braided Majorana fermions that can be accommodated in a multi-particle Hilbert space;

Class b, untruncated ($t = -1$) L_∞ level: the N -particle energy eigenvalues E are

$$E = 0, 1, \dots, N \quad \text{for any given } N; \quad (32)$$

in this case there is no plateau and the maximal energy eigenvalues grow linearly with N .

A parametrization alternative to (25) consists in expressing t , belonging to the $|t| = 1$ unit circle, as

$$t = -e^{2i\pi g}, \quad \text{for real values } g \in [0, 1[. \quad (33)$$

The connection with the (25) parametrization of [7] is given by

$$f = -2g - 1 \mod 2, \quad g = -\frac{1}{2}(f + 1) \mod 1. \quad (34)$$

The (33) parametrization simplifies the analysis of the level- s roots of unity; L_s and the L_∞ untruncated case are respectively specified by the following g values

$$\begin{aligned} L_s &: \text{recovered from } g = \frac{r}{s} & \text{with } r, s \text{ mutually prime integers,} \\ L_\infty &: \text{recovered from } g = 0. \end{aligned} \quad (35)$$

At the first orders the g values are

$$L_\infty : g_\infty = 0; \quad L_2 : g_2 = \frac{1}{2}; \quad L_3 : g_3 = \frac{1}{3}, \frac{2}{3}; \quad L_4 : g_4 = \frac{1}{4}, \frac{3}{4}; \quad L_5 : g_5 = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}. \quad (36)$$

Since the physics only depends on the s level and not on a given particular representative, without loss of generality we can set $r = 1$ and express the inequivalent s levels as

$$L_s : \text{recovered from } g_s = \frac{1}{s}. \quad (37)$$

The $g = 0$ case of the untruncated L_∞ level is recovered in the limit

$$g_\infty = \lim_{s \rightarrow \infty} g_s = 0. \quad (38)$$

The t roots of unity which correspond to (37) are

$$t_s = e^{\pi i (\frac{2}{s} - 1)}. \quad (39)$$

4 A quantum group derivation of the truncations

The results of [5] (see also the discussion in [7]) led to an open question. The truncations of the multi-particle spectra at roots of unity are reminiscent of the well-known special features of the quantum groups representations at roots of unity, see [15] and [16]. On the other hand the approach of [5] (which is based on [6]) does not directly use quantum group data. The compatible braided tensor product is applied to the $\mathfrak{gl}(1|1)$ superalgebra, not its quantum counterpart.

The construction which allows to recover braided multi-particle quantizations of Majorana qubits from quantum group reps is presented here.

It should be mentioned that the naive expectation that one should directly work with the quantum superalgebra $\mathcal{U}_q(\mathfrak{gl}(1|1))$ is not viable (this quantum superalgebra and its representations have been investigated in [14, 17–20]). The reason is simple. The creation operator γ entering (2) is nilpotent and the same is true for its $\mathcal{U}_q(\mathfrak{gl}(1|1))$ quantum group counterpart. Due to the homomorphism of the coproduct, we get the nilpotent quantum group expression $\Delta_q(\gamma)^2 = \Delta_q(\gamma) \cdot \Delta_q(\gamma) = \Delta_q(\gamma^2) = 0$ for the $\mathcal{U}_q(\mathfrak{gl}(1|1))$ coproduct. On the other hand, a nonvanishing $(\Delta(\gamma))^2 \neq 0$ coproduct induced by the braiding, see formula (20), is essential to produce the multi-particle spectra of the braided Majorana qubits.

Clearly, some other construction has to be done.

The solution is found by working within the quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$, inducing the multi-particle states by applying its coproduct to a specific representation and, furthermore, implementing a consistent superselection of the energy spectra. These steps allow to recover the multi-particle spectra of the braided Majorana qubits.

4.1 The $\mathfrak{osp}(1|2)$ superalgebra and its quantum extension

We introduce here, following [21] (see also [22]) the quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$ as a deformation of the $\mathfrak{osp}(1|2)$ Lie superalgebra. The complex deformation parameter is denoted, as in [21], with η while the [23] conventions for the $\mathfrak{osp}(1|2)$ generators are adopted. We can identify q as $q = e^\eta$.

The 5-generator $\mathfrak{osp}(1|2)$ Lie superalgebra is defined by the nonvanishing (anti)commutators

$$\begin{aligned} [H, F_\pm] &= \pm \frac{1}{2} F_\pm, & [H, E_\pm] &= \pm E_\pm, \\ [E_+, E_-] &= 2H, & [E_\pm, F_\mp] &= -F_\pm, \\ \{F_+, F_-\} &= \frac{1}{2} H, & \{F_\pm, F_\pm\} &= \pm \frac{1}{2} E_\pm. \end{aligned} \quad (40)$$

The two odd generators $F_\pm$ are the fermionic simple roots, while the three even generators $H, E_\pm$ define the $\mathfrak{sl}(2)$ subalgebra; H is the Cartan generator.

The second-order Casimir invariant C_2 is given by

$$C_2 = H^2 + \frac{1}{2}(E_+ E_- + E_- E_+) - (F_+ F_- - F_- F_+); \quad (41)$$

C_2 commutes with any generator of $\mathfrak{osp}(1|2)$, so that $[C_2, z] = 0$ for any $z = H, F_\pm, E_\pm$.

The $\mathfrak{osp}(1|2)$ superalgebra possesses the graded anti-involution $*$, given by

$$\begin{aligned} H^* &= H, & E_\pm^* &= E_\pm, & F_\pm^* &= \pm F_\mp, \\ \text{so that } (z^*)^* &= (-1)^{\varepsilon_z} z, & \text{with } \varepsilon_H, \varepsilon_{E_\pm} &= 0 \text{ and } \varepsilon_{F_\pm} &= 1. \end{aligned} \quad (42)$$

The quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$ is generated, see [21], by the three elements $H, F_\pm$ satisfying, in terms of the complex parameter $\eta \neq 0$, the (anti)commutation relations

$$\begin{aligned} [H, F_\pm]_\eta &= \pm \frac{1}{2} F_\pm, \\ \{F_+, F_-\}_\eta &= \frac{\sinh(\eta H)}{\sinh(2\eta)} = \frac{e^{\eta H} - e^{-\eta H}}{e^{2\eta} - e^{-2\eta}}. \end{aligned} \quad (43)$$

In the limit when η goes to zero one recovers the ordinary $\mathfrak{osp}(1|2)$ anticommutator $\{F_+, F_-\}$:

$$\lim_{\eta \rightarrow 0} \{F_+, F_-\}_\eta = \{F_+, F_-\} = \frac{1}{2} H. \quad (44)$$

The quantum superalgebra $\mathcal{U}_q(\mathfrak{osp}(1|2))$ has the structure of a graded Hopf superalgebra where, in particular, the following relations for the coproduct hold:

$$\begin{aligned} \Delta(H) &= H \otimes \mathbf{1} + \mathbf{1} \otimes H, \\ \Delta(F_\pm) &= F_\pm \otimes e^{\frac{\eta}{2}H} + e^{-\frac{\eta}{2}H} \otimes F_\pm. \end{aligned} \quad (45)$$

The tensor product is $\mathbb{Z}_2$ -graded and declared to satisfy, for any set of elements, the relation

$$(a \otimes b) \cdot (c \otimes d) = (-1)^{\varepsilon_b \varepsilon_c} ac \otimes bd \quad \text{for } \varepsilon_{b,c} = 0, 1, \quad (46)$$

where the ε grading of the generators is given in (42).

Due to the coassociativity, the $\Delta^{(2)}(F_+)$ coproduct which belongs to the tensor product of three spaces is given by

$$\Delta^{(2)}(F_+) = F_+ \otimes e^{\frac{\eta}{2}H} \otimes e^{\frac{\eta}{2}H} + e^{-\frac{\eta}{2}H} \otimes F_+ \otimes e^{\frac{\eta}{2}H} + e^{-\frac{\eta}{2}H} \otimes e^{-\frac{\eta}{2}H} \otimes F_+. \quad (47)$$

A single-particle Hilbert space $\mathcal{H}_\eta$ can be expressed as a lowest-weight representation of $\mathcal{U}_q(\mathfrak{osp}(1|2))$, defined by the Fock vacuum $|0\rangle_\eta$ such that

$$\begin{aligned} H|0\rangle_\eta &= \lambda|0\rangle_\eta, \\ F_-|0\rangle_\eta &= 0, \end{aligned} \quad (48)$$

where λ is a given “vacuum energy”.

The Hilbert space $\mathcal{H}_\eta$ is spanned by the (possibly infinite) series of vectors $|n\rangle_\eta$, given by

$$|n\rangle_\eta = F_+^n |0\rangle_\eta, \quad \text{where } n = 0, 1, 2, 3, \dots \quad (49)$$

It follows from (43) that a non-vanishing vector $|n\rangle_\eta$ is an eigenvector of H with $\lambda + \frac{n}{2}$ eigenvalue:

$$H|n\rangle_\eta = (\lambda + \frac{n}{2})|n\rangle_\eta. \quad (50)$$

The representations of the quantum group $\mathcal{U}_q(\mathfrak{osp}(1|2))$ and the conditions on their finiteness (which, in our conventions, depend on the pair of values λ, η) have been investigated in [21, 22, 24]. It is not necessary to present these conditions here since we have a very special purpose; we are interested in introducing the multi-particle Hilbert spaces induced by $\mathcal{U}_q(\mathfrak{osp}(1|2))$ which reproduce the multi-particle braided Majorana qubits Hilbert spaces. This implies that the single-particle Hilbert spaces have to be spanned by two vectors, $|0\rangle$ and $|1\rangle$ with respective energy eigenvalues 0 and 1. This implies at first that we have to set

$$\lambda = 0 \quad \text{and} \quad H_1 = 2H \quad \text{to be the normalized single-particle Hamiltonian.} \quad (51)$$

Since no finite two-dimensional Hilbert space is recovered from $\mathcal{U}_q(\mathfrak{osp}(1|2))$, a suitable two-dimensional finite Hilbert space $\mathcal{H}_\eta^{(1)}$ is constructed from $\mathcal{H}_\eta$ by applying a projection.

Let us therefore introduce the $P^2 = P$ projector $P = \text{diag}(1, 1, 0, 0, 0 \dots)$, defined as

$$P|0\rangle_\eta = |0\rangle_\eta, \quad P|1\rangle_\eta = |1\rangle_\eta, \quad \text{while } P|n\rangle_\eta = 0 \quad \text{for } n \geq 2. \quad (52)$$

We can consistently set the two-dimensional, single-particle Hilbert space being given by

$$\mathcal{H}_\eta^{(1)} \subset \mathcal{H}_\eta, \quad \text{spanned by the } P \text{ eigenvectors with } +1 \text{ eigenvalue, so that } |0\rangle_\eta, |1\rangle_\eta \in \mathcal{H}_\eta^{(1)}. \quad (53)$$

The projected single-particle Hamiltonian is expressed as PH_1P .

In the multi-particle sectors, the two-particle Hilbert space $\mathcal{H}_\eta^{(2)}$ is spanned by the vectors

$$(P \otimes P) \cdot \widehat{\Delta(F_+^n)} \cdot (|0\rangle_\eta \otimes |0\rangle_\eta) \in \mathcal{H}_\eta^{(2)} \quad (54)$$

(as usual in our conventions, the hat on the coproduct symbol indicates that it is evaluated in the given representation). It is easily shown that, due to the P projection, the integer n can at most take the values $n = 0, 1, 2$.

The $(N + 1)$ -particle Hilbert space $\mathcal{H}_\eta^{(N+1)}$ is spanned by the vectors

$$(P \otimes P \otimes \dots \otimes P) \cdot \widehat{\Delta^{(N)}(F_+^n)} \cdot (|0\rangle_\eta \otimes |0\rangle_\eta \otimes \dots \otimes |0\rangle_\eta), \quad (55)$$

where the $\widehat{\Delta^{(N)}}$ coproduct acts on the tensor product of $N + 1$ spaces. Due to the P projection, a simple combinatorics shows that n is restricted, for generic values of the deformation parameter η , to be given by $n = 0, 1, 2, \dots, N + 1$.

4.2 Recovering the roots of unity truncations

The roots of unity truncations for the projected N -particle $\mathcal{H}_\eta^{(N)}$ Hilbert spaces are recovered by mimicking the construction presented in Section 2 for the braided Majorana qubits. For this reason we introduce the “building blocks” of the N -particle creation operators $\widehat{\Delta^{(N-1)}(F_+)}$. Since no confusion will arise, we use the same symbols as the ones previously introduced in (17). There are two reasons for that: an unnecessarily burdening of the notation is avoided and, furthermore, it is easier to make the connection with the relevant formulas, such as (18), needed for the derivation of the truncations.

The new definition of the building blocks is

$$\begin{aligned} \text{For } N = 2 : \quad A_1^\dagger &:= F_+ \otimes e^{\frac{\eta}{2}H}, \quad A_2^\dagger := e^{-\frac{\eta}{2}H} \otimes F_+. \\ \text{For } N = 3 : \quad B_1^\dagger &:= F_+ \otimes e^{\frac{\eta}{2}H} \otimes e^{\frac{\eta}{2}H}, B_2^\dagger := e^{-\frac{\eta}{2}H} \otimes F_+ \otimes e^{\frac{\eta}{2}H}, B_3^\dagger := e^{-\frac{\eta}{2}H} \otimes e^{-\frac{\eta}{2}H} \otimes F_+. \end{aligned} \quad (56)$$

Extending the new definition to the N -particle building blocks $A_{k;N}^\dagger$ such as those entering (21) is immediate.

Contrary to their (17) counterparts, the (56) building blocks $A_i^\dagger, B_j^\dagger$ are not nilpotent: $(A_i^\dagger)^2, (B_j^\dagger)^2 \neq 0$. On the other hand, the contributions of their squares to the multi-particle sectors is killed by applying the projector P defined in (52).

By taking into account that $e^{uH}F_+e^{-uH} = e^{\frac{u}{2}}F_+$ for a complex parameter u , the analogous of the (18) formulas now read

$$\begin{aligned} A_2^\dagger \cdot A_1^\dagger &= -e^{-\frac{\eta}{2}} A_1^\dagger \cdot A_2^\dagger & \text{and} \\ B_2^\dagger \cdot B_1^\dagger &= -e^{-\frac{\eta}{2}} B_1^\dagger \cdot B_2^\dagger, & B_3^\dagger \cdot B_1^\dagger = -e^{-\frac{\eta}{2}} B_1^\dagger \cdot B_3^\dagger, & B_3^\dagger \cdot B_2^\dagger = -e^{-\frac{\eta}{2}} B_2^\dagger \cdot B_3^\dagger, \end{aligned} \quad (57)$$

with the identification

$$t = e^{-\frac{\eta}{2}}. \quad (58)$$

For the 2-particle sector, the $t = 1$ truncation of the $n = 2$ energy level is recovered from

$$\widehat{\Delta(F_+^2)} = (1-t)A_1^\dagger A_2^\dagger + (\dots) \Rightarrow \widehat{\Delta(F_+^2)}|_{t=1} \equiv 0, \quad (59)$$

where $(\dots)$ denotes the extra-terms killed by the P projection. We therefore reobtain the level-2 root of unity.

A straightforward computation for the 3-particle sector shows that the contribution to the $n = 3$ energy level is expressed by the formula

$$\widehat{\Delta(F_+^3)} = -t^{-\frac{3}{2}}(t^3 - 2t^2 + 2t - 1)(F_+e^{-\eta H} \otimes F_+ \otimes F_+e^{\eta H}) + (\dots); \quad (60)$$

once more, $(\dots)$ denotes the extra-terms killed by the P projection.

The truncation of the 3-particle $n = 3$ energy level is implied by

$$\widehat{\Delta(F_+^3)} \equiv 0 \Rightarrow t^3 - 2t^2 + 2t - 1 = 0. \quad (61)$$

Besides the $t = 1$ level-2 solution, the two level-3 roots solving the above cubic equations are $t = e^{\pm \frac{\pi i}{3}}$.

In terms of the (33) parametrization $t = -e^{2\pi i g}$ we can set from (58):

$$\eta = -2\pi i(2g - 1). \quad (62)$$

By identifying the level- s roots as $g_s = \frac{1}{s}$ from (37) and $t_s = e^{\pi i(\frac{2}{s}-1)}$ from (39) we get, in particular,

$$\begin{aligned} g = \frac{1}{2} &\Rightarrow \eta_2 = 0, & t_2 &= 1, \\ g = \frac{1}{3} &\Rightarrow \eta_3 = \frac{2\pi i}{3}, & t_3 &= e^{-\frac{\pi i}{3}}, \\ g = 0 &\Rightarrow \eta_\infty = 2\pi i, & t_\infty &= -1. \end{aligned} \quad (63)$$

where the last formula corresponds to the untruncated case.

The level-3 root corresponds to a third root of unity of the quantum group derivation via the identifications

$$(-t_3)^3 = 1 \quad \text{and} \quad q_3 = e^{\eta_3} \Rightarrow q_3^3 = 1. \quad (64)$$

The extension of the analysis to generic n energy levels for the N -particle sectors is immediate. It follows from the equivalence of the building blocks commutation relations such as (56) for $N = 2, 3$, with the (17) commutation relations under the identification $t = e^{-\frac{\eta}{2}}$.

5 Braided tensor products via intertwining operators

In this Section we present an alternative description (with respect to the one based on [5] and discussed in Section 2) of the multi-particle quantization of the braided Majorana qubits. Following [6], the $\otimes_{br}$ tensor product which enters the Section 2 construction is consistently *declared* to be braided. On the other hand, in many applications it would be useful to realize this construction via an ordinary tensor product $\otimes$ (the braiding being introduced via suitably defined intertwining operators). This approach is presented here. Specifically, we replace the (17,21) building blocks entering the multi-particle creation operators with new building blocks constructed in terms of the intertwining operators. The key issue is to reproduce the nilpotency conditions (19) and the (18,22) commutation relations. These relations are necessary and sufficient to obtain the multi-particle spectra of the braided Majorana qubits.

As already done in the previous Section, in order not to burden the notation and make easier the contact with the results of Section 2, the same symbols will be used for the new building blocks; no confusion will arise because it is clearly specified which is which.

We introduce here a minimal realization of the intertwining operators which are realized by 2×2 constant matrices (a non-minimal realization which is applied to the third root of unity truncation is presented in Appendix B).

The braided tensor product $\otimes_{br}$ applied to the $\gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ creation operator introduced in (2) satisfies the relation (4) which, with the (5) position for B_t , gives

$$(\mathbb{I}_2 \otimes_{br} \gamma) \cdot (\gamma \otimes_{br} \mathbb{I}_2) = -t(\gamma \otimes_{br} \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes_{br} \gamma) = -t(\gamma \otimes_{br} \gamma). \quad (65)$$

In terms of the (33) parametrization we set $t = -e^{2i\pi g}$.

By introducing a suitably defined intertwining operator W_t , the above braiding relation can be expressed in terms of an ordinary tensor product $\otimes$ through the positions

$$(\gamma \otimes_{br} \mathbb{I}_2) \mapsto \gamma \otimes \mathbb{I}_2, \quad (\mathbb{I}_2 \otimes_{br} \gamma) \mapsto W_t \otimes \gamma. \quad (66)$$

The mappings

$$\begin{aligned} (\mathbb{I}_2 \otimes_{br} \gamma) \cdot (\gamma \otimes_{br} \mathbb{I}_2) &\mapsto (W_t \otimes \gamma) \cdot (\gamma \otimes \mathbb{I}_2) = (W_t \gamma) \otimes \gamma \\ (\gamma \otimes_{br} \mathbb{I}_2) \cdot (\mathbb{I}_2 \otimes_{br} \gamma) &\mapsto (\gamma \otimes \mathbb{I}_2) \cdot (W_t \otimes \gamma) = (\gamma W_t) \otimes \gamma \end{aligned} \quad (67)$$

imply that the consistency condition for the 2×2 interwining operator W_t is given by

$$W_t \gamma = (-t) \gamma W_t. \quad (68)$$

A solution, expressed in terms of the $t = -e^{2i\pi g}$ position, is

$$W_t = \cos(-\pi g) \cdot \mathbb{I}_2 + i \sin(-\pi g) \cdot X, \quad \text{where } X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (69)$$

Indeed, we get

$$W_t \gamma = e^{2\pi i g} \gamma W_t. \quad (70)$$

The new building blocks of the braided 2-particle ($2P$) and 3-particle ($3P$) creation operators can be respectively defined as

$$\begin{aligned} 2P &: A_1^\dagger := \gamma \otimes \mathbb{I}_2, & A_2^\dagger &:= W_t \otimes \gamma; \\ 3P &: B_1^\dagger := \gamma \otimes \mathbb{I}_2 \otimes \mathbb{I}_2, & B_2^\dagger &:= W_t \otimes \gamma \otimes \mathbb{I}_2, & B_3^\dagger &:= W_t \otimes W_t \otimes \gamma. \end{aligned} \quad (71)$$

They are nilpotent, due to the nilpotency of the γ matrix:

$$(A_i^\dagger)^2 = 0 \quad \text{for } i = 1, 2 \quad \text{and} \quad (B_j^\dagger)^2 = 0 \quad \text{for } j = 1, 2, 3. \quad (72)$$

Their respective braiding relations are

$$(A_2^\dagger \cdot A_1^\dagger) = e^{2\pi ig} (A_1^\dagger \cdot A_2^\dagger) \quad (73)$$

and

$$(B_3^\dagger \cdot B_1^\dagger) = e^{2\pi ig} (B_1^\dagger \cdot B_3^\dagger), \quad (B_3^\dagger \cdot B_2^\dagger) = e^{2\pi ig} (B_2^\dagger \cdot B_3^\dagger), \quad (B_2^\dagger \cdot B_1^\dagger) = e^{2\pi ig} (B_1^\dagger \cdot B_2^\dagger). \quad (74)$$

The extension to the braiding of creation operators for $N > 3$ particle sectors is straightforward. It requires $N - 1$ tensor products and produces $2^N \times 2^N$ matrices; we can define

$$A_{k;N}^\dagger := \underbrace{W_t \otimes \cdots \otimes W_t}_{k-1} \otimes \gamma \otimes \underbrace{\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2}_{N-k} \quad \text{for } k = 1, 2, \dots, N. \quad (75)$$

The N -particle vacuum state is expressed by the 2^N column vector $|vac\rangle_N$ given by

$$|vac\rangle_N^T = (1, 0, 0, \dots, 0), \quad \text{where only the first entry is nonvanishing.} \quad (76)$$

By construction, the new (75) building blocks $A_{k;N}^\dagger$ are nilpotent ($(A_{k;N}^\dagger)^2 = 0$) and satisfy, for $k < k'$, the relations $A_{k';N}^\dagger A_{k;N}^\dagger = e^{2\pi ig} A_{k;N}^\dagger A_{k';N}^\dagger$.

The N -particle Hamiltonians H_N coincide with the matrices defined in (14).

It should be pointed out, for later convenience, that the $A_k^\dagger, A_k$ building blocks of the 2-particle creation and annihilation operators belong to a non-standard (i.e., non block-diagonal, see [25]) odd sector of a $\mathbb{Z}_2$ -graded decomposition of the 4×4 matrices. In the non-standard decomposition of the 4×4 matrices the nonvanishing entries (denoted with the symbol “*”) of the even (odd) sector M_0 (M_1) are accommodated in

$$M_0 \equiv \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{pmatrix}, \quad M_1 \equiv \begin{pmatrix} 0 & * & * & 0 \\ * & 0 & 0 & * \\ * & 0 & 0 & * \\ 0 & * & * & 0 \end{pmatrix}. \quad (77)$$

The $\mathbb{Z}_2$ grading is respected since

$$M_i \cdot M_j' = M_{i+j}'' \quad \text{for } i, j = 0, 1, \quad \text{with } i + j = 0, 1 \mod 2. \quad (78)$$

From (71) the 2-particle creation building blocks $A_1^\dagger, A_2^\dagger$ and their respective conjugate matrices A_1, A_2 are

$$\begin{aligned} A_1^\dagger &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & A_2^\dagger &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{-i\pi g} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\pi g} & 0 \end{pmatrix}, \\ A_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & A_2 &= \begin{pmatrix} 0 & e^{i\pi g} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-i\pi g} \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (79)$$

An even 4×4 central charge c is defined as

$$c = \text{diag}(1, 1, 1, 1). \quad (80)$$

As shown in Section 8, the 4 matrices in (79) and the central charge c close a generalized “mixed-bracket” Heisenberg-Lie algebra.

6 Indistinguishability as a superselection

Let us consider the braided set (75) of N -particle creation operators $A_{k;N}^\dagger$ ($k = 1, 2, \dots, N$); the operator $A_{k;N}^\dagger$ produces, when applied to the (76) vacuum state $|vac\rangle_N$, an excited state in the k -th position. Two scenarios can be considered. In the first scenario the particles in the k -th and k' -th ($k' \neq k$) positions are assumed to be of different type; in the alternative second scenario they are of the same type. The latter case brings the notion of quantum indistinguishability which implies, following the discussion of Section 2, that the powers of a unique particle creation operator have to be applied. The analogous of formula (21) for the (75) building blocks constructed in terms of the intertwining operator W_t reads as

$$A^{(N)\dagger} := \sum_{k=1}^N A_{k;N}^\dagger. \quad (81)$$

The truncation phenomenon discussed in Section 2 for the root-of-unity braidings is implied by the $(A^{(N)\dagger})^n$ powers of this composite operator. The indistinguishability case can be recovered from the first scenario by applying a superselection of the observables. This feature is easily illustrated in the 2-particle setting for the third-root-of-unity. It is immediate to generalize this construction to arbitrary N -particle sectors and arbitrary values of the g parameter defining, under the $t = -e^{2i\pi g}$ identification, W_t in formula (69).

For the third root of unity of the 2-particle sector we specialize the two 4×4 creation operators $A_1^\dagger, A_2^\dagger$ from (79) as given by $g = 1/3$. By setting

$$\overline{A}_1^\dagger = A_{1|g=\frac{1}{3}}^\dagger, \quad \overline{A}_2^\dagger = A_{2|g=\frac{1}{3}}^\dagger, \quad (82)$$

we obtain

$$\overline{A}_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \overline{A}_2^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ e^{-i\pi/3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\pi/3} & 0 \end{pmatrix}. \quad (83)$$

These creation operators satisfy

$$\overline{A}_2^\dagger \cdot \overline{A}_1^\dagger = j \overline{A}_1^\dagger \cdot \overline{A}_2^\dagger \quad \text{for } j = e^{i\frac{2}{3}\pi}. \quad (84)$$

The four basis vectors spanning the Hilbert space $\mathcal{H}_{dist}$ of 2 distinct particles can be given by

$$v_{00}, \quad v_{10} = \overline{A}_1^\dagger v_{00}, \quad v_{01} = \overline{A}_2^\dagger v_{00}, \quad v_{11} = \overline{A}_1^\dagger \overline{A}_2^\dagger v_{00}, \quad \text{where } v_{00}^T = (1, 0, 0, 0). \quad (85)$$

The two-particle Hamiltonian is $H_2 = \text{diag}(0, 1, 1, 2)$.

The associated Hilbert space of indistinguishable particles is the 3-dimensional Hilbert subspace $\mathcal{H}_{ind} \subset \mathcal{H}_{dist}$ spanned by the following v_E energy eigenvectors with respective eigenvalues $E = 0, 1, 2$:

$$v_{E=0} = v_{00}, \quad v_{E=1} = \overline{A}^\dagger v_{00}, \quad v_{E=2} = (\overline{A}^\dagger)^2 v_{00}, \quad \text{for } \overline{A}^\dagger = \overline{A}_1^\dagger + \overline{A}_2^\dagger, \quad (86)$$

so that

$$v_{E=0} = v_{00}, \quad v_{E=1} = v_{10} + v_{01}, \quad v_{E=2} = -j^2 v_{11}. \quad (87)$$

The indistinguishable Hilbert space $\mathcal{H}_{ind}$ is recovered from the superselection induced by the Hermitian projector operator $\bar{P}$ given by

$$\bar{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}j & 0 \\ 0 & -\frac{1}{2}j^2 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where } \bar{P}^2 = \bar{P} \quad \text{and} \quad \bar{P}^\dagger = \bar{P}. \quad (88)$$

The projector $\bar{P}$ commutes with the Hamiltonian H_2 :

$$[\bar{P}, H_2] = 0. \quad (89)$$

The superselected Hilbert subspace spanned by the +1 eigenvectors of $\bar{P}$ coincides with the Hilbert space $\mathcal{H}_{ind}$. Indeed, we get

$$\bar{P} \cdot v_{E=0,1,2} = v_{E=0,1,2}, \quad \text{while} \quad \bar{P} \cdot (v_{10} - v_{01}) = 0. \quad (90)$$

7 About Volichenko algebras and metasymmetries

The notion of “symmetries wider than supersymmetry” was stressed in the Leites-Serganova [8] paper and their further [9] work. A key point that they mentioned is the possibility of statistics-changing maps which do not preserve the grading of ordinary superalgebras.

As an implementation of this scheme they introduced the notion of *Volichenko algebras* (after a theorem proved by Volichenko) as nonhomogeneous subspaces of Lie superalgebras closed under the superbracket. In [8] the possible application to parastatistics was pointed out; the concept of superspace was enlarged to *metaspace* to take into account the class of transformations which do not preserve the grading. Similarly, the concept of symmetry or supersymmetry was enlarged to that of “metasymmetry”.

Technically, a Volichenko algebra satisfies a condition which is known as “metaabelianess”; this means that any triple of x, y, z generators satisfies the identity

$$[x, [y, z]] = 0 \quad (91)$$

where, in the above relations, the ordinary commutators are taken.

An intriguing example of a class of Volichenko algebras was presented in [9]. Denoted as $\mathfrak{vgl}_\mu(p|q)$, they are given by the $\mathfrak{gl}(\mathfrak{p}|q)$ space of $(p+q) \times (p+q)$ supermatrices. Under the $\mathbb{Z}_2$ -decomposition $\mathfrak{gl}(\mathfrak{p}|q) = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ into even/odd sectors, by fixing $\mu = a/b \in \mathbb{C}P^1$, a multiplication $\mathfrak{h}_1 \times \mathfrak{h}_1 \rightarrow \mathfrak{h}_0$ is introduced through the formula

$$(x, y)_\mu = a[x, y] + b\{x, y\} \quad \text{for any } x, y \in \mathfrak{h}_1. \quad (92)$$

For $ab \neq 0$ the $(\cdot, \cdot)_\mu$ bracket is an interpolation of ordinary commutators and anticommutators.

This particular example was my starting point to explore the possibility of introducing “mixed brackets” to describe the parastatistics of the braided Majorana qubits. There were two main reasons for that:

- i) a single-particle Majorana qubit is created/annihilated by the γ, β generators in the odd sector of the $\mathfrak{gl}(1|1)$ superalgebra, as shown in formulas (2, 3);
- ii) the building blocks of the 2-particle creation/annihilation operators are accommodated in

the odd sector of a $\mathfrak{gl}(2|2)$ superalgebra, see formula (79), under the (77) non-standard decomposition into even/odd matrices.

These considerations lead to the introduction of “mixed-bracket” generalized Heisenberg-Lie algebras which close generalized Jacobi identities (such new algebras are presented in Section 8). To my surprise, these algebras do not satisfy, as it can be easily checked, the metaabelianess condition (91) for any triple of generators; therefore, they are not Volichenko algebras according to the Leites-Serganova’s definition. Despite of that, they present a closed mixed-bracket structure; furthermore, as shown in Section 10, the generators of the generalized Heisenberg-Lie algebras define a mixed-bracket closed dynamical symmetry of a matrix Partial Differential Equation.

This leads to a relevant consideration which will be further discussed in the Conclusions. It points out towards a relaxation of the Leites-Serganova’s notion of “metasymmetry” which allows to accommodate mixed-bracket structures that do not necessarily satisfy the Volichenko metaabelianess condition. A quote from the original [8] paper seems to provide the solution: *“This feature of Volichenko algebras could be significant for parastatistics because once we abandon the Bose-Fermi statistics there seem to be too many ad hoc ways to generalize”*.

As a matter of fact, the mathematical structure of a simple model like the one here investigated (the multi-particle braided Majorana qubits) implies a parastatistics which has all key ingredients: mixed brackets, transformations and symmetries which do not preserve the $\mathbb{Z}_2$ -grading of superalgebras, etc.; for this model it is the dynamics of the braided multi-particle sector which forces the introduction of a parastatistics which does not fulfill the metaabelianess condition. This means that the mathematical structure is not obtained from an ad hoc prescription, but it is implied by the multi-particle dynamics of the model. It is shown in Section 8 that the mixed-bracket Heisenberg-Lie algebras satisfy a new identity which can be regarded as a metaabelianess condition expressed in terms of the mixed brackets.

Concerning Volichenko algebras, historical motivations and references leading to their introduction are found in [8, 9]. A construction of Volichenko algebras as algebras of differential operators is found in [26]. A recent account with updated references on Volichenko algebras is [27].

8 Braided qubits and mixed-bracket Heisenberg-Lie algebras

As mentioned in the previous Section, the Leites-Serganova notion of “symmetries wider than supersymmetry” applies to statistics-changing maps; it is the source of inspiration for introducing a mixed-bracket structure for the multi-particle creation/annihilation operators of the braided Majorana qubits. The construction, which is here presented, implies a mixed bracket generalization of the Heisenberg-Lie algebras.

Let X, Y be two operators. Their mixed-bracket, defined in terms of a ϑ_{XY} angle and denoted as $(X, Y)_{\vartheta_{XY}}$, is an interpolation of the ordinary $[X, Y]$ commutator and $\{X, Y\}$ anti-commutator. We can set

$$(X, Y)_{\vartheta_{XY}} := i \sin(\vartheta_{XY}) \cdot [X, Y] + \cos(\vartheta_{XY}) \cdot \{X, Y\}, \quad (93)$$

where ϑ_{XY} belongs, *mod* 2π , to the interval $\vartheta_{XY} \in [-\pi, \pi[$.

The following identity holds:

$$(Y, X)_{\vartheta_{YX}} = (X, Y)_{\vartheta_{XY}} \quad \text{for} \quad \vartheta_{YX} = -\vartheta_{XY}. \quad (94)$$

With the above definitions, if X, Y are Hermitian operators, the right hand side is also Hermitian:

$$X^\dagger = X, \quad Y^\dagger = Y \quad \rightarrow \quad (X, Y)_{\vartheta_{XY}}^\dagger = (Y, X)_{\vartheta_{YX}} = (X, Y)_{\vartheta_{XY}}. \quad (95)$$

We are now in the position to introduce the generalized, mixed-bracket, Heisenberg-Lie algebras of level- s for the 2-particle and 3-particle sectors. Due to the coassociativity of the (9) coproduct, the extension to the induced mixed-bracket generalized Heisenberg-Lie algebras for any N -particle sector is immediate.

Remark 1: the 2-particle level- s generalized Heisenberg-Lie algebra consists of the 5 generators entering formulas (79) and (80) with the position $g = \frac{1}{s}$. The 5 generators are the 4×4 matrices $A_1, A_2, A_1^\dagger, A_2^\dagger$ and the central charge $c = \text{diag}(1, 1, 1, 1)$.

Remark 2: the 3-particle level- s generalized Heisenberg-Lie algebra admits 7 generators which are presented as 8×8 matrices. They are the three creation operators $B_1^\dagger, B_2^\dagger, B_3^\dagger$ introduced, with the position $g = \frac{1}{s}$, in the second line of formula (71), together with their respective adjoint operators B_1, B_2, B_3 ($B_i = (B_i^\dagger)^\dagger$ for $i = 1, 2, 3$) and the central charge $c_3 = \text{diag}(1, 1, 1, 1, 1, 1, 1, 1)$.

At a formal level, the 2-particle and 3-particle mixed-bracket Heisenberg-Lie algebras can be recovered from, respectively, 2-oscillator and 3-oscillator Heisenberg-Lie algebras by replacing the ordinary (anti)commutators with the mixed brackets, so that $[\cdot, \cdot] \rightarrow (\cdot, \cdot)$ for appropriate choices of the angles.

Let us now specify the angles and introduce the closed algebraic structures.

It is convenient, for the 2-particle sector, to rename the generators as

$$G_0 := c = \text{diag}(1, 1, 1, 1), \quad G_{+1} := A_1^\dagger, \quad G_{-1} := A_1, \quad G_{+2} := A_2^\dagger, \quad G_{-2} := A_2. \quad (96)$$

In principle a suffix s should be introduced to specify the level of the truncation; it will be omitted in order not to burden the notation.

The Hermiticity conditions are

$$G_0^\dagger = G_0, \quad G_{\pm 1}^\dagger = G_{\mp 1}, \quad G_{\pm 2}^\dagger = G_{\mp 2}. \quad (97)$$

The $\mathbb{Z}_2$ -grading (77) of the 4×4 matrices implies that the generator G_0 is even ($\varepsilon = 0$), while the four generators $G_{\pm 1}, G_{\pm 2}$ are odd ($\varepsilon = 1$).

For $I, J = 0, \pm 1, \pm 2$, a suitable choice of the ϑ_{IJ} angles giving a closed structure to the $(G_I, G_J)_{\vartheta_{IJ}} = i \sin(\vartheta_{IJ}) \cdot [G_I, G_J] + \cos(\vartheta_{IJ}) \cdot \{G_I, G_J\}$ brackets is the following.

The vanishing brackets involving the central element G_0 coincide with an ordinary commutator; for any $I = 0, \pm 1, \pm 2$ we have

$$(G_0, G_I)_{\pm \frac{\pi}{2}} = (G_I, G_0)_{\mp \frac{\pi}{2}} = 0 \quad \Leftrightarrow \quad [G_0, G_I] = [G_I, G_0] = 0. \quad (98)$$

The extra angles entering the $(G_I, G_J)_{\vartheta_{IJ}}$ brackets for $I, J = \pm 1, \pm 2$ are determined by the following construction. Let us introduce the N^L, N^R diagonal operators

$$N^L = -\frac{1}{2} \cdot \text{diag}(1, 1, -1, -1), \quad N^R = -\frac{1}{2} \cdot \text{diag}(1, -1, 1, -1) \quad (99)$$

such that

$$[N^L, G_I] = \lambda_I^L G_I, \quad [N^R, G_I] = \lambda_I^R G_I, \quad (100)$$

where

$$\lambda_{\pm 1}^L = \pm 1, \quad \lambda_{\pm 2}^L = 0, \quad \lambda_{\pm 1}^R = 0, \quad \lambda_{\pm 2}^R = \pm 1. \quad (101)$$

The further parameters μ_I, ν_I are introduced through the positions

$$\mu_I = \lambda_I^L + \lambda_I^R, \quad \nu_I = (\lambda_I^L)^2 - (\lambda_I^R)^2. \quad (102)$$

Their corresponding values for the G_I generators are read from the table

	μ_I	ν_I
$G_{+1} :$	+1	+1
$G_{-1} :$	-1	+1
$G_{+2} :$	+1	-1
$G_{-2} :$	-1	-1

(103)

At the level- s ($s = 2, 3, 4, 5, \dots$), the angles ϑ_{IJ} for $I, J = \pm 1, \pm 2$ are determined by the formula

$$\vartheta_{IJ} = k_s \cdot \mu_I \mu_J \cdot (\nu_I - \nu_J) \quad \text{for} \quad k_s = \frac{s+2}{4s} \pi. \quad (104)$$

As mentioned before, this construction formally resembles a 2-oscillator algebra with the only nonvanishing brackets given by

$$(G_{\pm 1}, G_{\mp 1}) = (G_{\pm 2}, G_{\mp 2}) = G_0. \quad (105)$$

The mixed-bracket formulas, with the explicit insertion of the ϑ_{IJ} angle dependence, are

$$\begin{aligned} (G_{+1}, G_{-1})_0 &= (G_{-1}, G_{+1})_0 = G_0, & (G_{+1}, G_{+1})_0 &= (G_{-1}, G_{-1})_0 = 0, \\ (G_{+2}, G_{-2})_0 &= (G_{-2}, G_{+2})_0 = G_0, & (G_{+2}, G_{+2})_0 &= (G_{-2}, G_{-2})_0 = 0, \end{aligned} \quad (106)$$

together with

$$\begin{aligned} (G_{+1}, G_{+2})_{+\frac{s+2}{2s}\pi} &= (G_{+2}, G_{+1})_{-\frac{s+2}{2s}\pi} = 0, \\ (G_{+1}, G_{-2})_{-\frac{s+2}{2s}\pi} &= (G_{-2}, G_{+1})_{+\frac{s+2}{2s}\pi} = 0, \\ (G_{-1}, G_{+2})_{-\frac{s+2}{2s}\pi} &= (G_{+2}, G_{-1})_{+\frac{s+2}{2s}\pi} = 0, \\ (G_{-1}, G_{-2})_{+\frac{s+2}{2s}\pi} &= (G_{-2}, G_{-1})_{-\frac{s+2}{2s}\pi} = 0. \end{aligned} \quad (107)$$

The brackets entering (106) are ordinary anticommutators, while the brackets expressed by (107) are, for $s > 2$, a linear combination of commutators and anticommutators.

At $s = 2$ the (98,106,107) brackets define an ordinary, 2 fermionic oscillators, Heisenberg-Lie algebra.

For any given $s = 3, 4, 5, \dots$, the (98,106,107) brackets define a closed, 5-generator, mixed-bracket algebra which will be denoted as the “2-particle, mixed-bracket Heisenberg-Lie algebra (of level- s)”.

Since G_0 is a central element, these level- s mixed-bracket algebras not only satisfy generalized Jacobi identities; they satisfy the stronger identity

$$(G_I, (G_J, G_K)) = 0 \quad \text{for any } I, J, K = 0, \pm 1, \pm 2. \quad (108)$$

By expressing the angle dependence, the above identity reads as

$$(G_I, (G_J, G_K)_{\vartheta_{JK}})_{\vartheta_{I, J+K}} = 0, \quad (109)$$

where G_0 , the only possible nonvanishing element for (G_J, G_K) , is recovered from $J + K = 0$.

One should compare the similarity of the expression (109) with the metaabelianess condition (91). The (91) condition involves only ordinary commutators, while the (109) identity involves the $(.,.)$ mixed brackets. It is quite tempting to relax, as discussed in Section 7, the metaabelianess condition presented in [8] with a new metaabelianess criterium based on the identity (109) which involves mixed brackets.

One can indeed easily check that, while the set of 5 generators G_I for $I = 0, \pm 1, \pm 2$ always satisfy (109), some combinations of the operators violate the (91) metaabelianess condition. A violation is given, e.g., by the expression

$$[G_{+1}, [G_{+2}, G_{-2}]] = 2i \sin\left(\frac{\pi}{s}\right) (E_{21} - E_{43}) \neq 0, \quad (110)$$

where E_{ij} denotes the 4×4 matrix with entry 1 at the intersection of the i^{th} row with the j^{th} column and 0 otherwise.

We present, for completeness, the 3-particle sector of the level- s mixed-bracket Heisenberg-Lie algebras. As done for the 2-particle case, it is convenient to rename the 7 generators which were previously introduced in the above *Remark 2*. Let us set

$$\overline{G}_0 := c_3 = \text{diag}(1, 1, 1, 1, 1, 1, 1), \quad \overline{G}_{+i} := B_i^\dagger, \quad \overline{G}_{-i} := B_i, \quad \text{for } i = 1, 2, 3. \quad (111)$$

The brackets defined for $\overline{G}_0$ are ordinary commutators:

$$(\overline{G}_0, \overline{G}_I)_{\pm \frac{\pi}{2}} = (\overline{G}_I, \overline{G}_0)_{\mp \frac{\pi}{2}} = 0, \quad \text{for } I = 0, \pm 1, \pm 2, \pm 3. \quad (112)$$

The following brackets, defined for $i = 1, 2, 3$, are ordinary anticommutators:

$$(\overline{G}_{+i}, \overline{G}_{-i})_0 = (\overline{G}_{-i}, \overline{G}_{+i})_0 = \overline{G}_0, \quad (\overline{G}_{+i}, \overline{G}_{+i})_0 = (\overline{G}_{-i}, \overline{G}_{-i})_0 = 0. \quad (113)$$

There are three sets (S_{12}, S_{13}, S_{23}) of mixed brackets, respectively involving the generators:

$$S_{12} : \overline{G}_{\pm 1} \text{ and } \overline{G}_{\pm 2}; \quad S_{13} : \overline{G}_{\pm 1} \text{ and } \overline{G}_{\pm 3}; \quad S_{23} : \overline{G}_{\pm 2} \text{ and } \overline{G}_{\pm 3}. \quad (114)$$

The respective mixed brackets are recovered, *with the same values of the angles*, from fomula (107) under the positions

$$\begin{aligned} S_{12} &: G_{\pm 1} \mapsto \overline{G}_{\pm 1}, & G_{\pm 2} \mapsto \overline{G}_{\pm 2}, \\ S_{13} &: G_{\pm 1} \mapsto \overline{G}_{\pm 1}, & G_{\pm 2} \mapsto \overline{G}_{\pm 3}, \\ S_{23} &: G_{\pm 1} \mapsto \overline{G}_{\pm 2}, & G_{\pm 2} \mapsto \overline{G}_{\pm 3}. \end{aligned} \quad (115)$$

The three 5-generator subalgebras, respectively spanned by the sets of generators $\{\overline{G}_0, \overline{G}_{\pm 1}, \overline{G}_{\pm 2}\}$, $\{\overline{G}_0, \overline{G}_{\pm 1}, \overline{G}_{\pm 3}\}$ and $\{\overline{G}_0, \overline{G}_{\pm 2}, \overline{G}_{\pm 3}\}$, are all isomorphic to the level- s , 2-particle, mixed-bracket Heisenberg-Lie algebras.

As mentioned before, the extension of the construction to N -particle mixed-bracket Heisenberg-Lie algebras for any integer $N > 3$ is immediate.

9 Color parafermionic oscillators from the $s \rightarrow \infty$ limit

The level- s mixed-bracket Heisenberg-Lie algebras provide an interpolation between the ordinary $s = 2$ fermionic oscillators and the untruncated spectrum case recovered in the $s \rightarrow \infty$ limit. Since the $s = \infty$ spectrum is untruncated one can loosely refer, with a caveat, to this case as “bosonic”. The caveat is that in both truncated and untruncated cases the original $\mathbb{Z}_2$ -grading holds; the creation operators are odd and the spectrum is split into even (including the vacuum) and odd states. The superselection rule which holds for ordinary bosons/fermions is still in place: therefore one can only makes superpositions (linear combinations) either of even states or of odd states (without mixing them). The most general state in the $\mathbb{Z}_2$ -graded Hilbert space is either $\varphi = \sum_a c_a \varphi_a$ (the sum being taken on the orthonormal basis of the even states) or $\psi = \sum_\alpha c_\alpha \psi_\alpha$ (the sum being taken on the orthonormal basis of the odd states). This point was discussed at length in [5].

For a single Majorana qubit this implies that the $\mathbb{Z}_2$ -graded Hilbert space is $\mathbb{C}^{1|1}$, instead of $\mathbb{C}^2$ of an ordinary qubit. The analogous of the Bloch sphere for a Majorana qubit is the “ $\mathbb{Z}_2$ -sphere” expressed by two points respectively identified with $+1$ or -1 . The space of the physically inequivalent configurations individuated by the ray vectors corresponds to 1 bit of information. We have an even state (labeled as “0” in a suitable parametrization) associated with the vacuum ray vector and an odd state (labeled as “1”) associated with the excited Majorana ray vector.

On the basis of the (32) result presented in Section 3, the $\mathbb{Z}_2$ -graded Hilbert spaces of the N -particle sectors are given, in the $s \rightarrow \infty$ untruncated limit, by

$$\begin{aligned} \mathbb{C}^{n+1|n+1}, & \quad \text{for an odd integer } N = 2n + 1 \quad \text{and} \\ \mathbb{C}^{n+1|n}, & \quad \text{for an even integer } N = 2n. \end{aligned} \quad (116)$$

In the $s \rightarrow \infty$ limit the brackets recovered from formulas (98,106,107) in the 2-particle sector and from formulas (112,113,115) in the 3-particle sector are ordinary commutators and anticommutators (that is, there are no mixed terms since either the sin function or the cos function entering the right hand side of (93) is vanishing). Despite being ordinary commutators/anticommutators, the brackets are *arranged in a peculiar way*. To illustrate this point we present the 2-particle case.

In the $s \rightarrow \infty$ limit the five 4×4 matrices G_I ($I = 0, \pm 1, \pm 2$) are given by

$$\begin{aligned} G_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & G_{+1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & G_{+2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\ G_{-1} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & G_{-2} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (117)$$

Both sets of $\{G_0, G_{\pm 1}\}$ and $\{G_0, G_{\pm 2}\}$ subalgebras close a fermionic oscillator algebra:

$$\begin{aligned} [G_0, G_{\pm 1}] &= 0, & \{G_{\pm 1}, G_{\pm 1}\} &= 0, & \{G_{+1}, G_{-1}\} &= G_0, \\ [G_0, G_{\pm 2}] &= 0, & \{G_{\pm 2}, G_{\pm 2}\} &= 0, & \{G_{+2}, G_{-2}\} &= G_0. \end{aligned} \quad (118)$$

Unlike the $s = 2$ case, whose full algebra corresponds to 2 fermionic oscillators, in the $s \rightarrow \infty$ limit the brackets taken for one $G_{\pm 1}$ and one $G_{\pm 2}$ generator are given by ordinary commutators:

$$[G_{\pm 1}, G_{\pm 2}] = 0, \quad [G_{\pm 1}, G_{\mp 2}] = 0. \quad (119)$$

The (118,119) brackets define the $\mathbb{Z}_2^2$ -graded parafermionic oscillators algebra.

Let's see in more detail the $\mathbb{Z}_2^2$ -graded construction. $\mathbb{Z}_2^2$ -graded “color” Lie algebras and superalgebras were introduced by Rittenberg-Wyler in [28, 29] (see also [30]) as extensions of ordinary Lie algebras and superalgebras. The generators of a color (super)algebra $\mathfrak{g}$ are accommodated in a $\mathbb{Z}_2^2$ grading:

$$\mathfrak{g} = \mathfrak{g}_{00} \oplus \mathfrak{g}_{10} \oplus \mathfrak{g}_{01} \oplus \mathfrak{g}_{11}. \quad (120)$$

A $[\cdot, \cdot]$ bracket which respects the grading is introduced:

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \quad \text{is such that} \quad [\mathfrak{g}_{ij}, \mathfrak{g}_{kl}] \subset \mathfrak{g}_{i+k, j+l} \pmod{2}. \quad (121)$$

The $[\cdot, \cdot]$ bracket is either a commutator or an anticommutator (one can therefore express $[\mathfrak{g}_{ij}, \mathfrak{g}_{kl}] \equiv \mathfrak{g}_{ij} \cdot \mathfrak{g}_{kl} \pm \mathfrak{g}_{kl} \cdot \mathfrak{g}_{ij}$). The color (super)algebra satisfies a $\mathbb{Z}_2^2$ -graded Jacobi identity (see [28, 29] for details). The two consistent assignments of commutators/anticommutators for the $[\cdot, \cdot]$ bracket respectively define a $\mathbb{Z}_2^2$ -graded color Lie algebra or a $\mathbb{Z}_2^2$ -graded color Lie superalgebra. The case under consideration here is that of the color superalgebra. Its (anti)commutators assignment can be read from the entries of the table:

	00	10	01	11
00	0	0	0	0
10	0	1	0	1
01	0	0	1	1
11	0	1	1	0

(122)

The entry 0 denotes a commutator ($0 \equiv [\cdot, \cdot]$); the entry 1 denotes an anticommutator ($1 \equiv \{\cdot, \cdot\}$). One could note that the generators of the 10-sector anticommute among themselves; the same is true for the generators belonging to the 01-sector. Particles associated with these sectors satisfy the Pauli exclusion principle and are therefore fermionic-like in nature. On the other hand, unlike ordinary fermions, 10-particles commute with 01-particles. This means that they

define two classes of parafermions. Similarly, the bosonic-like particles belonging to the 11-sector *anticommute* with the 10- and 01-parafermions: the particles of the 11-sector are parabosons.

The 5-generator closed algebraic structure defined by the (118,119) brackets is a $\mathbb{Z}_2^2$ -graded color Lie superalgebra. It will be denoted as $\mathfrak{h}_{\text{pf}}(2)$, standing for “2 parafermionic oscillators Heisenberg-Lie algebra”. The assignments of its generators into the $\mathbb{Z}_2^2$ -graded sectors are

$$G_0 \in \mathfrak{h}_{\text{pf}}(2)_{00}, \quad G_{\pm 1} \in \mathfrak{h}_{\text{pf}}(2)_{10}, \quad G_{\pm 2} \in \mathfrak{h}_{\text{pf}}(2)_{01}, \quad \emptyset \in \mathfrak{h}_{\text{pf}}(2)_{11}, \quad (123)$$

the color Lie superalgebra 11-graded sector being empty.

The composite generator of the Universal Enveloping Algebra given by

$$G_{+3} := G_{+1} \cdot G_{+2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (124)$$

belongs to the 11-graded sector.

A $\mathbb{Z}_2^2$ grading of the 4×4 matrices is introduced as follows (the possible nonvanishing entries in each sector are denoted with the symbol “*”):

$$M_{00} \equiv \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}, \quad M_{10} \equiv \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix}, \quad M_{01} \equiv \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix}, \quad M_{11} \equiv \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}. \quad (125)$$

A comment is in order: the above $\mathbb{Z}_2^2$ grading of the matrices implies that the Hilbert space recovered from the action of the G_{+1}, G_{+2}, G_{+3} matrices on the 2-particle vacuum state $v_{00} = (1, 0, 0, 0)^T$ can be endowed with a $\mathbb{C}^{1|1|1|1}$ grading. The superselected indistinguishable Hilbert space, as described in Section 6, is not compatible with the second $\mathbb{Z}_2$ -graded structure, but only with the original $\mathbb{Z}_2$ grading expressed in formula (77). As a consequence, the 2-particle indistinguishable Hilbert space is three-dimensional and given, in accordance with the second formula of (116) for $n = 1$, by $\mathbb{C}^{2|1}$.

Extra gradings can be introduced not only in the $s \rightarrow \infty$ limit, but also in the truncated cases. An example is presented in Appendix B for $s = 3$ (the third root of unity truncation). The introduction of a $\mathbb{Z}_3$ grading is made possible by realizing the gradings of the creation/annihilation operators via a nonminimal matrix representation.

In the $s \rightarrow \infty$ limit the construction of the corresponding parafermionic oscillators algebras, for any other N -particle sector with $N \geq 3$, is immediate.

10 Dynamical metasymmetries of the mixed-bracket Heisenberg-Lie algebras

The mixed-bracket generalizations of the Heisenberg-Lie algebras introduced in Section 8 appear as dynamical symmetries of Ordinary Differential Equations given by Matrix Schrödinger equations in $0 + 1$ dimensions. This property is easily illustrated for the simplest nontrivial example which corresponds to the 2-particle Hamiltonian H_2 . The extension to N -particle Hamiltonians with $N > 2$ is immediate.

As recalled in Section 8, the mixed-bracket Heisenberg-Lie algebras are not metaabelian according to the original [8, 9] definition; nevertheless, they are generalized algebras which do not respect the even/odd $\mathbb{Z}_2$ -grading of Lie superalgebras and which satisfy an alternative metaabelian condition given by the (109) identity. Acting as dynamical symmetries of an ordinary Matrix Differential Equation, they deserve being referred to as *dynamical metasymmetries*.

The Matrix Schrödinger equation under consideration is

$$(i\partial_t \cdot \mathbb{I}_4 - H_2)\Psi(t) = 0, \quad (126)$$

where $H_2 = \text{diag}(0, 1, 1, 2)$ and $\Psi(t)$ is a 4-component vector.

The $\Psi_{ij}(t)$ solutions of (126) can be expressed in terms of the creation $A_1^\dagger, A_2^\dagger$ and annihilation A_1, A_2 operators introduced in (71,79) and defined for the given angle πg . The further position $g = \frac{1}{s}$ for $s = 2, 3, 4, 5, \dots$ defines the level- s truncation.

We have

$$\begin{aligned} \Psi_{00}(t) &= v_{00}, & \text{where } v_{00}^T &= (1, 0, 0, 0), \\ \Psi_{10}(t) &= e^{-it} A_1^\dagger v_{00} = e^{-it} v_{10}, & \text{where } v_{10}^T &= (0, 0, 1, 0), \\ \Psi_{01}(t) &= e^{-it} A_2^\dagger v_{00} = e^{-it} v_{01}, & \text{where } v_{01}^T &= (0, e^{i\pi g}, 0, 0), \\ \Psi_{11}(t) &= e^{-2it} A_1^\dagger A_2^\dagger v_{00} = e^{-2it} v_{11}, & \text{where } v_{11}^T &= (0, 0, 0, e^{i\pi g}). \end{aligned} \quad (127)$$

Due to the commutators

$$[H_2, A_1^\dagger] = A_1^\dagger, \quad [H_2, A_2^\dagger] = A_2^\dagger, \quad [H_2, A_1] = -A_1, \quad [H_2, A_2] = -A_2, \quad (128)$$

it is immediate that, by setting

$$S_1^\dagger = e^{-it} A_1^\dagger, \quad S_2^\dagger = e^{-it} A_2^\dagger, \quad S_1 = e^{it} A_1, \quad S_2 = e^{it} A_2, \quad (129)$$

one ends up with four plus one symmetry operators (the extra operator being the 4×4 identity operator $c := \mathbb{I}_4$) satisfying

$$[S_\sharp, i\partial_t \cdot \mathbb{I}_4 - H_2] = 0 \quad \text{for } S_\sharp = S_1^\dagger, S_2^\dagger, S_1, S_2, c. \quad (130)$$

If $\Psi(t)$ is a solution of the matrix Schrödinger equation (126), then $\Psi'(t) = S_\sharp \Psi(t)$ is also (provided that $\Psi'(t) \neq 0$) a solution of the same equation.

Due to the fact that the $e^{\pm it}$ phases play no role as far as the brackets are concerned, the five operators $S_1^\dagger, S_2^\dagger, S_1, S_2, c$ produce the same mixed-bracket Heisenberg-Lie algebra expressed by the formulas (98,106,107). Up to the phases, the identification is $c \equiv G_0, S_1^\dagger \equiv G_{+1}, S_1 \equiv G_{-1}, S_2^\dagger \equiv G_{+2}, S_2 \equiv G_{-2}$. The simplest nontrivial case corresponds to the level-3 root of unity obtained from $g = \frac{1}{3}$. For this value the $A_1^\dagger, A_2^\dagger$ matrices are those expressed in formulas (82,83).

A dynamical metasymmetry is also produced in the $s \rightarrow \infty$ limit; in this case the symmetry is given by the $\mathbb{Z}_2^2$ -graded parafermionic oscillator algebra $\mathfrak{h}_{\text{pf}}(2)$ whose brackets are the (anti)commutators (118,119).

It is worth pointing out that this is not the first example of a $\mathbb{Z}_2^2$ -graded dynamical symmetry. It was shown in [31,32] that the symmetry operators of the Partial Differential Equations of the nonrelativistic Lévy-Leblond spinors possess a $\mathbb{Z}_2^2$ -graded Lie superalgebra structure. This can be regarded, in a different context, as another implementation of the “symmetries wider than supersymmetry” concept.

11 Conclusions

Braided Majorana qubits have an obvious interest in the light of the Kitaev's proposal [1] to use them for encoding topological quantum computations which offer topological protection from quantum decoherence. Due to their neat combinatorics, the braided Majorana qubits are also an ideal playground to investigate different mathematical frameworks which can be applied to describe parastatistics. This is the content of the paper whose main results are here summarized.

The probably most relevant of such frameworks is based on the Leites-Serganova [8,9] notion of “symmetries wider than supersymmetry”, concerning the existence of statistics-changing maps which do not preserve the $\mathbb{Z}_2$ -grading of ordinary Lie superalgebras. Possible applications to parastatistics (including the introduction of brackets which interpolate ordinary commutators and anticommutators) were suggested in the [8,9] mathematical papers. On the other hand this mathematical structure, as far as I know, mostly passed unnoticed in the physical literature. In this paper it is shown that the multi-particle sectors of braided Majorana qubits are described by a new class of algebras, the generalized, mixed-bracket Heisenberg-Lie algebras introduced in Section 8.

Rather unexpectedly, the original Leites-Serganova construction has to be relaxed to accommodate a more general setting. In [8,9] the statistics-changing maps were induced by nonhomogeneous subspaces of Lie superalgebras closed under the superbracket. This leads to the notion of *Volichenko algebras* which satisfy the (91) metaabelianess condition (for any X, Y, Z triple of operators, the identity $[X, [Y, Z]] = 0$, which involves ordinary commutators, is satisfied). The mixed-bracket Heisenberg-Lie algebras are not metaabelian, but satisfy the (109) identity, which can be referred to as a “metaabelianess condition for the mixed brackets” (for any X, Y, Z triple of operators, the $(X, (Y, Z)) = 0$ identity is satisfied, where $(.,.)$ denotes the mixed brackets).

The Leites-Serganova notion of *metasymmetry*, as discussed in Section 10, can be applied to this more general setting. This is obviously quite a fascinating topic, both in pure mathematics and in connection with physics.

Concerning the other results presented in this paper, let me first remind that the traditional framework to introduce parastatistics is based, see [33,34], on the trilinear relations; more recently, the [6] approach based on graded Hopf algebras endowed with a braided tensor product was introduced (the connection between the two approaches has been investigated in [35,36]).

The [6] framework was applied in [37,38] to prove the theoretical detectability of the para-particles implied by the $\mathbb{Z}_2^2$ -graded “color” Lie (super)algebras introduced by Rittenberg-Wyler in [28,29]. Later on, the extension to a higher dimensional representation of the braid group was applied in [5] to first-quantize the multi-particle braided Majorana qubits. The truncations of the multi-particle spectra at roots of unity were observed. This obviously suggests, see [15,16], a quantum group interpretation of the truncations. On the other hand, a quantum group interpretation is not directly available from the [6] framework applied to the $\mathfrak{gl}(1|1)$ superalgebra endowed with a braided tensor product. The puzzle is solved in Section 4; the multi-particle spectra of the braided Majorana qubits are reconstructed from representations of the $\mathcal{U}_q(\mathfrak{osp}(1|2))$ quantum supergroup. The next topic, from Section 5, consists in realizing (via the introduction of intertwining operators) the *braided tensor products* of the creation/annihilation operators in terms of ordinary tensor products. This not only facilitates the computer-assisted algebraic manipulations (via Mathematica or any other favorite software); it lays the basis to construct the matrix representations which are needed in order to introduce the mixed-bracket Heisenberg-Lie

algebras.

The last topic worth mentioning is the $s \rightarrow \infty$ *untruncated* limit of the mixed-bracket Heisenberg-Lie algebras presented in Section 9. The algebraic structures recovered in the limit are (once more) the Rittenberg-Wyler color Lie superalgebras. In this limit the brackets are no longer interpolations of commutators/anticommutators, but they are reduced to ordinary (anti)commutators. Nevertheless, they are organized in a different way with respect to the ordinary (anti)commutators of the bosons/fermions statistics, see e.g. the entries in the (122) table. The $s \rightarrow \infty$ limit implies paraparticles obeying the types of parastatistics investigated in [37,38].

As a final remark, analyzing in retrospect the results of [31,32] (the $\mathbb{Z}_2^2$ -graded Lie superalgebra structure of the symmetry operators of the nonrelativistic Lévy-Leblond spinors), one can conclude (see the remark at the end of Section 10) that this is another example of the Leites-Serganova notion of “symmetries wider than supersymmetry”.

The results of this paper point towards a more general setting than the one investigated in [8,9] to accommodate statistics-changing maps; it can be applied to braid statistics recovered from quantum groups, parastatistics recovered from color Lie (super)algebras, etc.

Appendix A: a comment on mixed brackets and quons

It is worth mentioning that “mixed brackets” which interpolate commutators and anticommutators can be defined for other types of parastatistics. Perhaps the most notable example is the algebra of quons which was introduced by Greenberg [10,39] and Mohapatra [11]. Quons are q -deformed oscillators, defined for $-1 \leq q \leq 1$ (the constraint expressing the admissible quonic Hilbert spaces is discussed in [10]) which interpolate between fermions ($q = -1$) and bosons ($q = 1$). The “ q -mutators” of n creation/annihilation $a_i^\dagger, a_i$ quons, with $i = 1, 2, \dots, n$ are defined to satisfy

$$a_i a_j^\dagger - q a_i^\dagger a_j = \delta_{ij}. \quad (131)$$

Since its introduction, the quonic parastatistics has been intensely investigated; one can mention, e.g., the paper [40] relating quons to anyons.

It is a trivial exercise to express the q -mutator (131) of one ($n = 1$) quon as a mixed bracket, interpolating commutator and anticommutator. One has to set

$$a a^\dagger - q a^\dagger a = 1 \quad \Leftrightarrow \quad \cos^2(\theta_q) \cdot [a, a^\dagger] + \sin^2(\theta_q) \cdot \{a, a^\dagger\} = 1, \quad (132)$$

where the angle θ_q , comprised in the range $\theta_q \in [0, \pi/2]$, is given by

$$\theta_q = \arcsin \left(\sqrt{\frac{1-q}{2}} \right), \quad (q = 1 - 2 \sin^2 \theta_q). \quad (133)$$

Special θ_q angles are

- i) $\theta_q = 0$ for $q = 1$, which recovers the bosons,
- ii) $\theta_q = \frac{\pi}{4}$ for $q = 0$, which recovers the so-called, see [39], “infinite statistics” case,
- iii) $\theta_q = \frac{\pi}{2}$ for $q = -1$, which recovers the fermions.

The mixed-bracket Heisenberg-Lie algebras, which have been introduced in Section 8 and reproduce the multi-particle parastatistics of the braided Majorana qubits, differ from the quonic “mixed brackets” as derived in formulas (132,133). Indeed, the (75) building blocks $A_{k;N}^\dagger$ of the creation operators of the N -particle sector and their $A_{k;N} = (A_{k;N}^\dagger)^\dagger$ annihilation counterparts do not satisfy the quonic algebra (132) for any nontrivial $q \neq \pm 1$ value; it is easily shown that they satisfy, for any given $k = 1, 2, \dots, N$, an ordinary anticommutator:

$$A_{k;N} A_{k;N}^\dagger + A_{k;N}^\dagger A_{k;N} = \mathbb{I}_{2N}. \quad (134)$$

The genuine mixed brackets $(\cdot, \cdot)_\vartheta$ from (93) with a non-trivial angle dependence ($\vartheta \neq n\frac{\pi}{2}$ for integer values of n) are taken with respect to the creation/annihilation operators for $k \neq k'$, as shown, e.g., in formulas (107) for the 2-particle sector. At level- s ($s = 2, 3, 4, \dots$) we get, in particular,

$$(G_{+1}, G_{+2})_{+\frac{s+2}{2s}\pi} \equiv (A_{1;2}^\dagger, A_{2;2}^\dagger)_{+\frac{s+2}{2s}} = 0. \quad (135)$$

It should be stressed, as a final comment, that the parastatistics induced by the mixed-bracket Heisenberg-Lie algebras are not related to the quonic parastatistics.

Appendix B: about third-roots of unity and ternary algebras

In the minimal matrix representation, the N -particle sector of the braided Majorana qubits is realized, see (75), by $2^N \times 2^N$ matrices. Equivalent descriptions which produce isomorphic Hilbert spaces can be obtained from nonminimal representations. Let's consider the third root of unity; an example of a set of nonminimal representations is given by $2 \cdot 6^{N-1} \times 2 \cdot 6^{N-1}$ matrices. Unlike the minimal representations (75) with the special third root of unity case in (82, 83), this nonminimal set encodes a $\mathbb{Z}_3$ ternary grading. It can therefore find application to ternary physics, see for details [12] and references therein (one of such cases is the fractional supersymmetry discussed in [41]). Despite providing the same description of braided Majorana fermions, the minimal representation (75) at the third root of unity does not respect the $\mathbb{Z}_3$ grading and, therefore, cannot be used in these ternary physics applications.

The ternary construction of the braided Majorana qubits employs tensor products of the three 3×3 matrices Q_i (defined for $j = e^{i\frac{2}{3}\pi}$ with $j^3 = 1$) and their $Q_i^\dagger$ hermitian conjugates:

$$\begin{aligned} Q_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, & Q_2 &= \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix}, & Q_3 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\ Q_1^\dagger &= \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, & Q_2^\dagger &= \begin{pmatrix} 0 & 0 & j \\ j^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & Q_3^\dagger &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (136)$$

Following [12] a consistent $\mathbb{Z}_3$ grading can be assigned by setting, *mod* 3,

$$\deg(Q_i) = 1, \quad \deg(Q_i^\dagger) = 2, \quad \text{for } i = 1, 2, 3. \quad (137)$$

The non-minimal building blocks of the braided 2-particle creation/annihilation operators are

$$\tilde{A}_1^\dagger = \gamma \otimes \mathbb{I}_2 \otimes Q_1, \quad \tilde{A}_2^\dagger = \gamma \otimes \mathbb{I}_2 \otimes Q_2, \quad \tilde{A}_1 = \gamma^\dagger \otimes \mathbb{I}_2 \otimes Q_1^\dagger, \quad \tilde{A}_2 = \gamma^\dagger \otimes \mathbb{I}_2 \otimes Q_2^\dagger, \quad (138)$$

while for the 3-particle sector we have

$$\tilde{B}_1^\dagger = \gamma \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes Q_1 \otimes \mathbb{I}_3, \quad \tilde{B}_2^\dagger = \mathbb{I}_2 \otimes \gamma \otimes \mathbb{I}_2 \otimes Q_2 \otimes Q_1, \quad \tilde{B}_3^\dagger = \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \gamma \otimes Q_2 \otimes Q_2. \quad (139)$$

The nilpotent matrix γ , introduced in (2), is $\gamma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

The ternary 2-particle representation (138) is given by 12×12 matrices. The 3-particle ternary representation (139) with operators $B_i^\dagger$ and their hermitian conjugates is given by 72×72 matrices. The extension of the construction to the $N > 3$ particle sectors, producing $2 \cdot 6^{N-1} \times 2 \cdot 6^{N-1}$ matrices, is immediate.

The nilpotency of γ implies

$$(\tilde{A}_i^\dagger)^2 = 0 \quad \text{for } i = 1, 2 \quad \text{and} \quad (\tilde{B}_j^\dagger)^2 = 0 \quad \text{for } j = 1, 2, 3. \quad (140)$$

We further get the braiding relations

$$(\tilde{A}_2^\dagger \cdot \tilde{A}_1^\dagger) = j \cdot (\tilde{A}_1^\dagger \cdot \tilde{A}_2^\dagger) \quad \text{and} \quad (141)$$

$$(\tilde{B}_2^\dagger \tilde{B}_1^\dagger) = j \cdot (\tilde{B}_1^\dagger \tilde{B}_2^\dagger), \quad (\tilde{B}_3^\dagger \tilde{B}_1^\dagger) = j \cdot (\tilde{B}_1^\dagger \tilde{B}_3^\dagger), \quad (\tilde{B}_3^\dagger \tilde{B}_2^\dagger) = j \cdot (\tilde{B}_2^\dagger \tilde{B}_3^\dagger). \quad (142)$$

Formulas (140,141,142) reproduce, for $g = \frac{1}{3}$, the corresponding “minimal” formulas presented in (72,73,74).

In the non-minimal setting the quantization of the braided Majorana qubits at the third root of unity is reproduced by introducing the N -particle vacuum $|\widetilde{vac}\rangle_N$ as the $2 \cdot 6^{N-1}$ column vector $|\widetilde{vac}\rangle_N = e_1$, where e_1 denotes the column with first entry 1 and 0 otherwise.

For $N = 2, 3$ the Hamiltonians $\widetilde{H}_2, \widetilde{H}_3$ are respectively given by

$$\begin{aligned} \widetilde{H}_2 &= (\delta \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \delta) \otimes \mathbb{I}_3, \\ \widetilde{H}_3 &= (\delta \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \delta \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \delta) \otimes \mathbb{I}_3 \otimes \mathbb{I}_3. \end{aligned} \quad (143)$$

By introducing

$$\tilde{A}^\dagger = \tilde{A}_1^\dagger + \tilde{A}_2^\dagger, \quad \tilde{B}^\dagger = \tilde{B}_1^\dagger + \tilde{B}_2^\dagger + \tilde{B}_3^\dagger, \quad (144)$$

the 2, 3-particle Hilbert spaces $\widetilde{\mathcal{H}}_{(2)}, \widetilde{\mathcal{H}}_{(3)}$ are respectively spanned by the vectors

$$(\tilde{A}^\dagger)^n |\widetilde{vac}\rangle_2 \in \widetilde{\mathcal{H}}_{(2)}, \quad (\tilde{B}^\dagger)^n |\widetilde{vac}\rangle_3 \in \widetilde{\mathcal{H}}_{(3)}, \quad \text{for } n = 0, 1, 2, \dots \quad (145)$$

The truncation at the third root of unity is recovered from the identity

$$(\tilde{B}^\dagger)^3 = 0, \quad (146)$$

which implies that the 3-particle sector is spanned by the three energy eigenvectors $|\widetilde{vac}\rangle_3, \tilde{B}^\dagger |\widetilde{vac}\rangle_3$ and $(\tilde{B}^\dagger)^2 |\widetilde{vac}\rangle_3$ whose respective energy eigenvalues are $E = 0, 1, 2$.

The matrices in (138,139) reproduce the mixed brackets of, respectively, the 2-particle and 3-particle Heisenberg-Lie algebras at the level $s = 3$ (that is, the third root of unity). They induce the same algebras as the ones recovered from the corresponding minimal representations introduced in Section 5 (see, e.g., formula (71) specialized at the $g = \frac{1}{3}$ value).

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