

# Beyond the 10-fold way: 13 associative $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras

Zhanna Kuznetsova\* and Francesco Toppan†

December 7, 2021

\* *UFABC, Av. dos Estados 5001, Bangu, cep 09210-580, Santo André (SP), Brazil.*

† *CBPF, Rua Dr. Xavier Sigaud 150, Urca, cep 22290-180, Rio de Janeiro (RJ), Brazil.*

## Abstract

The “10-fold way” refers to the combined classification of the 3 associative division algebras (of real, complex and quaternionic numbers) and of the 7,  $\mathbb{Z}_2$ -graded, superdivision algebras (in a superdivision algebra each homogeneous element is invertible). The connection of the 10-fold way with the periodic table of topological insulators and superconductors is well known. Motivated by the recent interest in  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded physics (classical and quantum invariant models, parastatistics) we classify the associative  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras and show that 13 inequivalent cases have to be added to the 10-fold way. Our scheme is based on the “alphabetic presentation of Clifford algebras”, here extended to graded superdivision algebras. The generators are expressed as equal-length words in a 4-letter alphabet (the letters encode a basis of invertible  $2 \times 2$  real matrices and in each word the symbol of tensor product is skipped). The 13 inequivalent  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras are split into real series (4 subcases with 4 generators each), complex series (5 subcases with 8 generators) and quaternionic series (4 subcases with 16 generators).

CBPF-NF-004/21

---

\*E-mail: [zhanna.kuznetsova@ufabc.edu.br](mailto:zhanna.kuznetsova@ufabc.edu.br)

†E-mail: [toppan@cbpf.br](mailto:toppan@cbpf.br)

## 1 Introduction

Real numbers ( $\mathbb{R}$ ), complex numbers ( $\mathbb{C}$ ) and quaternions ( $\mathbb{H}$ ) are the 3 associative division algebras over the reals. By allowing a  $\mathbb{Z}_2$ -grading, 7 further associative superdivision algebras are encountered (in a graded superdivision algebra, each homogeneous element admits inverse). The total number of  $10 = 3 + 7$  is known as the “10-fold way”, see [1] for a short introduction on the topic of superdivision algebras.

This purely mathematical property has striking connections with physical applications. The 10-fold way appears in the construction of the so-called “periodic table of topological insulators and superconductors”, which presents features like the *mod* 8 Bott’s periodicity. The physical significance of the periodic table is discussed in [2]. In [3, 4] the Cartan’s classification of symmetric spaces was related to the 10 classes of symmetries of random-matrix theories. The 10-fold way classification of generic Hamiltonians and topological insulators and superconductors has been discussed in [5] (see also the references therein). On the mathematical side, different implementations of the 10-fold way have been analyzed in [6]; the one which is relevant for our paper (besides the Morita’s equivalence of Clifford modules and the graded extension of the Dyson’s threefold way [7]) is the direct relation with the [8] classification of graded Brauer groups, i.e. superdivision algebras.

Since there is no need to stop at  $\mathbb{Z}_2$ , the notion of superdivision algebra can be immediately extended to accommodate a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading. The aim of this paper is to classify the inequivalent, associative,  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras. We obtain 13 inequivalent cases. Before presenting the mathematical results we mention that the motivation of this work lies in the recent surge of interest in  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded physics. We present a succinct state of the art on this topic.  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie algebras and superalgebras were introduced in [9, 10] (see also [11]) and investigated by mathematicians ever since. The attempts to use them for physical applications were rather sporadic. This situation changed when it was pointed out in [12, 13] that  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebras naturally appear as dynamical symmetries of Partial Differential Equations describing Lévy-Leblond nonrelativistic spinors. An ongoing intense activity followed. A model of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded invariant quantum mechanics was discussed in [14], while the systematic construction of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded classical mechanics was introduced in [15] and the canonical quantization of the models presented in [16]. Further developments include the construction of two-dimensional models [17–19], graded superconformal quantum mechanics [20], superspace [18, 21, 22], extensions and bosonization of double-graded supersymmetric quantum mechanics [23, 24], and so forth. The construction of models with  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded parabosons was discussed in [18]. The theoretical detectability of the parafermions implied by the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebras was proved in [25], while in [26] the result was extended to the parabosons implied by  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie algebras (for previous works on  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded parastatistics see [27, 28] and the references therein). All this ongoing activity makes reasonable to expect that the classification of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras would not just be a mathematical curiosity, but could find a way to concrete physical applications.

As an efficient tool to classify  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras we apply the so-called “alphabetic presentation of Clifford algebras” introduced in [29], which is here extended to superdivision algebras. In this framework the generators are expressed as matrices defined by equal-length words in a 4-letter alphabet (the letters encode a basis of invertible  $2 \times 2$  real matrices and in each word the symbol of tensor product is skipped). Each one of the 7 inequivalent  $\mathbb{Z}_2$ -graded division algebra admits an alphabetic presentation; it follows from this property that any  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebra can be alphabetically presented. Within this scheme, spotting the inequivalent classes of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras

is reduced to a simple exercise in combinatorics. We obtain 13 inequivalent cases which are split into 4 real, 5 complex and 4 quaternionic subcases. The number of generators, in each subcase, is respectively 4, 8 and 16. Correspondingly, the faithful representations are given by  $4 \times 4$  matrices with real entries for the real subcases,  $8 \times 8$  matrices with real entries for the complex subcases and  $16 \times 16$  matrices with real entries for the quaternionic subcases.

For each inequivalent case the multiplication table of the superdivision algebras generators can be directly read from a given matrix representation.

The scheme of the paper is the following. The efficient tool of the “alphabetic presentation of Clifford algebras” to express real representations of Clifford algebras is briefly recalled in Section 2. In Section 3 the notion of graded superdivision algebra is explained. In Section 4 we apply the alphabetic framework to recover the 10-fold way classification of division and  $\mathbb{Z}_2$ -graded superdivision algebras. This paves the way for presenting, in Section 5, the core of the paper, namely the classification of the associative  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras. A brief commentary on the obtained results is given in the Conclusions. The alphabetic presentation of  $\mathbb{Z}_2$ -graded and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras is introduced in Appendix A.

## 2 The alphabetic presentation of Clifford algebras

It was pointed out in [30] that real, almost complex and quaternionic Clifford algebras are recovered from real matrix representations by taking into account the respective properties concerning the Schur’s lemma. This implies that, while almost complex and quaternionic Clifford algebras can be represented by complex matrices, nothing is lost when using only real matrices. The real representations discussed in [30] are particularly useful for classification purposes, since they allow to consider at once all three cases of real, complex and quaternionic structures.

The irreducible representations of Clifford algebras have been classified in [31]. It is well known, see e.g. [30], that Clifford algebra generators can be expressed as tensor products of the (complex) Pauli matrices and the  $2 \times 2$  identity. In order to get the real representations one should drop the imaginary unit  $i$ . It follows that real representations of Clifford algebras can be obtained by tensoring the 4 basis elements of the  $2 \times 2$  real matrices. One can associate a letter to each one of these 4 matrices. By dropping for convenience the unnecessary symbol “ $\otimes$ ” of tensor product, which is understood, a matrix representing a Clifford algebra generator can be expressed by a word in a 4-letter alphabet. This is the “alphabetic presentation of Clifford algebras” introduced in [29]. Any irreducible representation of a Clifford algebra is then expressed as a set of equal-length words written in this alphabet, the constructive algorithms presented in [30] and [32] being recasted [29] in this language.

We are now briefly detailing this construction since it will be applied in the following, at first to recover the 10-fold way and later to classify the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras.

One associates the four letters with a basis of invertible matrices spanning the vector space of  $2 \times 2$  real matrices. In this paper we use the following conventions:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (1)$$

The letter  $I$  is chosen because it stands for “the identity matrix”, while  $A$  stands for “the antisymmetric matrix”. An  $n$ -character word written in this 4-letter alphabet represents a  $2^n \times 2^n$  real matrix. There are some useful tips for detecting the matrix structure. For instance:

- i*) a word whose initial letter is  $I$  or  $X$  represents a block-diagonal matrix, while if it starts with  $Y$  or  $A$  the matrix is block-antidiagonal;
- ii*) a word containing an even (odd) number of letters  $A$  represents a symmetric (antisymmetric)

matrix;

iii) two different  $X, Y, A$  letters represent mutually anticommuting matrices, making easier to check whether two matrices defined by equal-length words either commute or anticommute.

For illustrative purposes and later convenience we discuss the alphabetic presentation of the quaternions. A faithful representation of the three imaginary quaternions  $q_i$  ( $i = 1, 2, 3$ ) satisfying

$$q_i q_j = -\delta_{ij} + \varepsilon_{ijk} q_k \quad (2)$$

(where  $\varepsilon_{ijk}$  is the totally antisymmetric tensor normalized as  $\varepsilon_{123} = 1$ ) is given by either

$$\bar{q}_1 = IA, \quad \bar{q}_2 = AY, \quad \bar{q}_3 = AX, \quad (3)$$

or by

$$\tilde{q}_1 = AI, \quad \tilde{q}_2 = YA, \quad \tilde{q}_3 = XA. \quad (4)$$

Following the rules mentioned above the  $4 \times 4$  real matrices expressed by (3) are given by

$$\bar{q}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \bar{q}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{q}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad (5)$$

The three imaginary quaternions  $\bar{q}_i$  defined by (3) satisfy the

$$\{\bar{q}_i, \bar{q}_j\} = -2\delta_{ij} \cdot \mathbb{I}_4 \quad (\mathbb{I}_4 := II) \quad (6)$$

relations, making them the gamma matrices (see below) of the  $Cl(0, 3)$  Clifford algebra. Their quaternionic structure is encoded, see [30], in the Schur's lemma, which states that the most general matrix  $S$  commuting with the  $\bar{q}_i$ 's is of quaternionic form, being expressed by

$$S = \lambda_0 \cdot II + \lambda_1 \cdot AI + \lambda_2 \cdot YA + \lambda_3 \cdot XA \quad \Rightarrow \quad [S, \bar{q}_i] = 0, \quad (7)$$

for arbitrary real values  $\lambda_J \in \mathbb{R}$ , with  $J = 0, 1, 2, 3$ .

$S$  is defined in terms of the  $\tilde{q}_j$  matrices entering (4). Obviously, the identification of (3) as defining the imaginary quaternions and of (4) as defining the associated quaternionic structure can be switched.

The multiplication table of the 4 letters, obtained from the (1) identifications, is

$r \backslash c$	$I$	$X$	$Y$	$A$
$I$	$I$	$X$	$Y$	$A$
$X$	$X$	$I$	$A$	$Y$
$Y$	$Y$	$-A$	$I$	$-X$
$A$	$A$	$-Y$	$X$	$-I$

(8)

The entries of the table denote the result of the multiplication of the row letters “ $r$ ” (acting on the left) with the column letters “ $c$ ”.

Our nomenclature of Clifford algebras follows [30, 32]. The Clifford algebra  $Cl(p, q)$  is the Enveloping algebra, over the  $\mathbb{R}$  field, of a set of  $n \times n$  gamma matrices  $\gamma_I$  ( $I = 1, 2, \dots, p + q$ ) satisfying, for any  $I, J$  pair:

$$\gamma_I \gamma_J + \gamma_J \gamma_I = 2\eta_{IJ} \cdot \mathbb{I}_n. \quad (9)$$

$\mathbb{I}_n$  denotes the  $n \times n$  identity matrix, while  $\eta_{IJ}$  is a pseudo-Euclidean diagonal metric with signature  $+1$  for  $I, J = 1, \dots, p$  and  $-1$  for  $I, J = p + 1, \dots, p + q$ .

The irreducible representation of  $Cl(p, q)$  is recovered when  $n$  is the minimal integer which allows solutions of (9) (sometimes it is useful, as in the applications to superdivision algebras, to skip the irreducibility requirement).

The three  $2 \times 2$  matrices associated with the letters  $X, Y, A$  are the gamma matrices of the  $Cl(2, 1)$  Clifford algebra; the whole set of four letters (1) is a two-dimensional faithful representation of  $\tilde{\mathbb{H}}$ , the algebra of the split-quaternions (see [33] for the definition of the split versions of the division algebras).

### 3 Graded superdivision algebras

In this Section we recall the notion of graded superdivision algebra. In a  $\mathbb{Z}_2$ -graded superdivision algebra the generators are split into even (belonging to the 0-graded sector) and odd (belonging to the 1-graded sector). In a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebra the generators are accommodated into 2 bits of information (00, 10, 01, 11 graded sectors). A graded superdivision algebra can be denoted as  $\mathbb{D}^{[p]}$ , where the non-negative integer  $p$  indicates the  $\mathbb{Z}_2^p$ -grading. In this work we limit to consider the  $p = 0, 1, 2$  values. The three ordinary associative division algebras over the reals ( $\mathbb{R}, \mathbb{C}, \mathbb{H}$ ) are obtained from  $p = 0$ . For later purposes we can set

$$\mathbb{R} := D_{\mathbb{R};1}^{[0]}, \quad \mathbb{C} := D_{\mathbb{R};2}^{[0]}, \quad \mathbb{H} := D_{\mathbb{R};3}^{[0]}. \quad (10)$$

For  $p = 1, 2$  we respectively have

$$\begin{aligned} \mathbb{D}^{[1]} &= \mathbb{D}_0^{[1]} \oplus \mathbb{D}_1^{[1]}, \\ \mathbb{D}^{[2]} &= \mathbb{D}_{00}^{[2]} \oplus \mathbb{D}_{01}^{[2]} \oplus \mathbb{D}_{10}^{[2]} \oplus \mathbb{D}_{11}^{[2]}. \end{aligned} \quad (11)$$

A homogeneous element  $g$  belongs to a given graded sector ( $g \in \mathbb{D}_i^{[1]}$  or  $g \in \mathbb{D}_{ij}^{[2]}$ , where  $i, j$  take values 0, 1). A multiplication, which respects the grading, is defined. Let  $g_A, g_B \in \mathbb{D}^{[p]}$  be two homogeneous elements, whose respective gradings are  $i_A, i_B$  for  $\mathbb{Z}_2$  and the pairs  $(i_A, j_A), (i_B, j_B)$  for  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . The multiplied element  $g_A \cdot g_B \in \mathbb{D}^{[p]}$  is homogeneous and its graded sector, either  $i_{A+B}$  or  $(i_{A+B}, j_{A+B})$ , is obtained from *mod 2* arithmetics:

$$\mathbb{Z}_2 : \quad i_{A+B} = i_A + i_B; \quad \mathbb{Z}_2 \times \mathbb{Z}_2 : \quad (i_{A+B}, j_{A+B}) = (i_A + i_B, j_A + j_B). \quad (12)$$

The unit element 1 will also be denoted as “ $e_0$ ” (it belongs to the 0 sector or, respectively, the 00 sector). In this paper we assume the multiplication to be associative.

In a graded superdivision algebra each nonvanishing homogeneous element is invertible. As a consequence, each graded sector is isomorphic, as vector space, to one of the three vector spaces of real numbers, complex numbers or quaternions. It easily follows that the common dimension (in real counting) of each graded sector of a given superdivision algebra is 1, 2 or 4, depending on the case.

Each graded sector is spanned by the respective set of basis vectors  $e_J, f_J, g_J, h_J$ , that can be assigned according to

$$\begin{aligned} \text{for } \mathbb{Z}_2 & : \quad e_J \in \mathbb{D}_0^{[1]}, \quad f_J \in \mathbb{D}_1^{[1]}, \\ \text{for } \mathbb{Z}_2 \otimes \mathbb{Z}_2 & : \quad e_J \in \mathbb{D}_{00}^{[2]}, \quad f_J \in \mathbb{D}_{01}^{[2]}, \quad g_J \in \mathbb{D}_{10}^{[2]}, \quad h_J \in \mathbb{D}_{11}^{[2]}. \end{aligned} \quad (13)$$

Depending on the spanning vector space, the suffix  $J$  is restricted to be

$$\mathbb{R} : J = 0; \quad \mathbb{C} : J = 0, 1; \quad \mathbb{H} : J = 0, 1, 2, 3. \quad (14)$$

A generic homogeneous element  $e$  belonging to either the 0 or the 00 graded sector is expressed by the linear combination  $e = \sum_J \lambda^J e_J$ , where the  $\lambda^J$ 's ( $\lambda^J \in \mathbb{R}$ ) are real parameters. This formula is extended in obvious way to homogeneous elements belonging to the other graded sectors. As mentioned before, in our conventions  $e_0$  will denote the unit element.

*Remark:* the following remark will be used later on. In a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebra the three graded sectors 01, 10, 11 are on equal footing and can be interchanged by permutation. Indeed, by setting  $\alpha = 01$ ,  $\beta = 10$ ,  $\gamma = 11$ , the table of the *mod 2* sums given in (12) of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading reads as follows:

	00	$\alpha$	$\beta$	$\gamma$
00	00	$\alpha$	$\beta$	$\gamma$
$\alpha$	$\alpha$	00	$\gamma$	$\beta$
$\beta$	$\beta$	$\gamma$	00	$\alpha$
$\gamma$	$\gamma$	$\beta$	$\alpha$	00

(15)

The table is invariant under permutations of  $\alpha, \beta, \gamma$ .

Each graded superdivision algebra over  $\mathbb{R}$  can be presented by normalizing any given generator  $g \in \mathbb{D}^{[p]}$  so that its square is, up to a sign, the identity ( $g^2 = \pm e_0$ ). This leaves a sign ambiguity ( $\pm g$ ) in the normalization of each generator  $g \neq e_0$ . The graded superdivision algebras are divided into classes of equivalence based on the following set of transformations:

- i)* the sign flippings  $g \mapsto \pm g$  for the generators  $g \neq e_0$ ,
- ii)* the permutation of the generators inside each graded sector and
- iii)* for the  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading ( $p = 2$ ), the permutation of the equal footing sectors 10, 01, 11.

This set of transformations defines the 7 (and respectively 13) inequivalent classes of  $\mathbb{Z}_2$ -graded ( $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded) superdivision algebras.

In each class of equivalence, a given multiplication table produced among generators induces equivalent multiplication tables obtained by applying the above three transformations. In the case of the quaternions, for instance, this is tantamount to flip the sign ( $\epsilon_{ijk} \mapsto -\epsilon_{ijk}$ ) of the totally antisymmetric tensor entering (2).

A multiplication table is straightforwardly recovered from a faithful matrix representation of the graded superdivision algebra.

## 4 The 10-fold way revisited

Each  $\mathbb{Z}_2$ -graded superdivision algebra admits an alphabetic presentation; the generators (see Appendix **A**) are expressed by equal-length words in a 4-letter alphabet and the (1) identification of letters and  $2 \times 2$  matrices holds.

In this Section we present the alphabetic derivation of the 10-fold way for division and  $\mathbb{Z}_2$ -graded superdivision algebras. We start at first with the three ordinary division algebras.

The field of the real numbers  $\mathbb{R}$  is the only exception with the (1) identification of letters with  $2 \times 2$  matrices, since the unit element  $e_0$  is the number 1; the real numbers can, nevertheless, be accommodated into the scheme by setting  $e_0 = I$ , the two-dimensional identity matrix.

The complex numbers  $\mathbb{C}$  are alphabetically expressed by the two single-character words  $I$  and  $A$  (respectively, the identity element and the imaginary unit).

The quaternions  $\mathbb{H}$  can be expressed by four two-character words: either  $II, IA, AY, AX$ , see (3) or  $II, AI, YA, XA$ , see (4). The  $4 \times 4$  matrices corresponding to these two choices are related by a similarity transformation.

In terms of the (10) positions we can express the sets of generators as

$$\begin{aligned} I &\in \mathbb{D}_{\mathbb{R};1}^{[0]}, \\ I, A &\in \mathbb{D}_{\mathbb{R};2}^{[0]}, \\ II, IA, AX, AY &\in \mathbb{D}_{\mathbb{R};3}^{[0]}. \end{aligned} \tag{16}$$

The seven  $\mathbb{Z}_2$ -graded superdivision algebras ( $7 = 2 + 3 + 2$ ) will be denoted as

$$\begin{aligned} \mathbb{D}_{\mathbb{R};*}^{[1]} &\text{ for the real series (with } * = 1, 2), \\ \mathbb{D}_{\mathbb{C};*}^{[1]} &\text{ for the complex series (with } * = 1, 2, 3) \text{ and} \\ \mathbb{D}_{\mathbb{H};*}^{[1]} &\text{ for the quaternionic series (with } * = 1, 2). \end{aligned}$$

#### 4.1 The real $\mathbb{Z}_2$ -graded superdivision algebras

Two inequivalent  $\mathbb{Z}_2$ -graded superdivision algebras are obtained from the real series. On the basis of formulas (A.3, A.5, A.7) in Appendix A they can be expressed as

$$\begin{aligned} \mathbb{D}_{\mathbb{R};1}^{[1]} &: \quad I \in \mathbb{D}_0^{[1]}, \quad A \in \mathbb{D}_1^{[1]}, \\ \mathbb{D}_{\mathbb{R};2}^{[1]} &: \quad I \in \mathbb{D}_0^{[1]}, \quad Y \in \mathbb{D}_1^{[1]}, \end{aligned} \tag{17}$$

where  $\mathbb{D}_0^{[1]}$  ( $\mathbb{D}_1^{[1]}$ ) denote the respective even and odd sectors.

Since  $A^2 = -I$ ,  $Y^2 = I$  we have that:

- i)  $\mathbb{D}_{\mathbb{R};1}^{[1]}$  corresponds to the complex numbers  $\mathbb{C}$  endowed with a  $\mathbb{Z}_2$ -grading;
- ii)  $\mathbb{D}_{\mathbb{R};2}^{[1]}$  corresponds to the split-complex numbers  $\tilde{\mathbb{C}}$  endowed with a  $\mathbb{Z}_2$ -grading.

#### 4.2 The complex $\mathbb{Z}_2$ -graded superdivision algebras

Three inequivalent  $\mathbb{Z}_2$ -graded superdivision algebras are obtained from the complex series. On the basis of formulas (A.3, A.5, A.7) in Appendix A they can be expressed as

$$\begin{aligned} \mathbb{D}_{\mathbb{C};1}^{[1]} &: \quad II, IA \in \mathbb{D}_0^{[1]}, \quad AX, AY \in \mathbb{D}_1^{[1]}, \\ \mathbb{D}_{\mathbb{C};2}^{[1]} &: \quad II, IA \in \mathbb{D}_0^{[1]}, \quad YX, YY \in \mathbb{D}_1^{[1]}, \\ \mathbb{D}_{\mathbb{C};3}^{[1]} &: \quad II, IA \in \mathbb{D}_0^{[1]}, \quad AI, AA \in \mathbb{D}_1^{[1]}, \end{aligned} \tag{18}$$

where  $\mathbb{D}_0^{[1]}$  ( $\mathbb{D}_1^{[1]}$ ) denote the respective even and odd sectors.

The alternative presentation  $II, IA \in \mathbb{D}_0^{[1]}$ ,  $YI, YA \in \mathbb{D}_1^{[1]}$  produces a superdivision algebra which is isomorphic to  $\mathbb{D}_{\mathbb{C};3}^{[1]}$ .

We have that:

- i)  $\mathbb{D}_{\mathbb{C};1}^{[1]}$  corresponds to a  $\mathbb{Z}_2$ -grading of the quaternions  $\mathbb{H}$ , realizing a graded representation of the  $Cl(0,3)$  Clifford algebra;
- ii)  $\mathbb{D}_{\mathbb{C};2}^{[1]}$  corresponds to a  $\mathbb{Z}_2$ -grading of the split-quaternions  $\tilde{\mathbb{H}}$ , realizing a  $4 \times 4$  graded matrix representation of the  $Cl(2,1)$  Clifford algebra;
- iii)  $\mathbb{D}_{\mathbb{C};3}^{[1]}$  corresponds to a  $\mathbb{Z}_2$ -grading of an algebra of commuting matrices.

### 4.3 The quaternionic $\mathbb{Z}_2$ -graded superdivision algebras

Two inequivalent  $\mathbb{Z}_2$ -graded superdivision algebras are obtained from the quaternionic series. On the basis of formulas (A.3, A.5, A.7) in Appendix **A** they can be expressed as

$$\begin{aligned} \mathbb{D}_{\mathbb{H};1}^{[1]} & : & III, IIA, IAY, IAX \in \mathbb{D}_0^{[1]}, & \quad AII, AIA, AAY, AAX \in \mathbb{D}_1^{[1]}, \\ \mathbb{D}_{\mathbb{H};2}^{[1]} & : & III, IIA, IAY, IAX \in \mathbb{D}_0^{[1]}, & \quad YII, YIA, YAY, YAX \in \mathbb{D}_1^{[1]}, \end{aligned} \quad (19)$$

where  $\mathbb{D}_0^{[1]}$  ( $\mathbb{D}_1^{[1]}$ ) denote the respective even and odd sectors.

The inequivalence of the two superdivision algebras given above is spotted by taking the squares of the odd generators. They produce, up to a sign, the  $8 \times 8$  identity matrix. The signs are  $(-+++)$  for  $\mathbb{D}_{\mathbb{H};1}^{[1]}$  and  $(+---)$  for  $\mathbb{D}_{\mathbb{H};2}^{[1]}$ .

In this Section we recovered, within the alphabetic presentation, the 7 superdivision algebras discussed in [1] and presented, under a different name, in [6].

## 5 The 13 inequivalent $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras

The alphabetic construction of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras has been discussed in Appendix **A**. We present here the results. We obtain 13 inequivalent cases (the defining classes of equivalence have been introduced in Section **3**). A representative, for each class of equivalence, is given below.

The 13 cases are split into  $13 = 4 + 5 + 4$  subcases; 4 subcases are obtained from the real series, 5 subcases from the complex series, 4 subcases from the quaternionic series. The 13  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras will be named as follows:

$$\begin{aligned} \mathbb{D}_{\mathbb{R};*}^{[2]} & \text{ for the real series (with } * = 1, 2, 3, 4), \\ \mathbb{D}_{\mathbb{C};*}^{[2]} & \text{ for the complex series (with } * = 1, 2, 3, 4, 5) \text{ and} \\ \mathbb{D}_{\mathbb{H};*}^{[2]} & \text{ for the quaternionic series (with } * = 1, 2, 3, 4). \end{aligned}$$

As mentioned in Appendix **A**, see formula (A.9), any  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebra can be characterized by its subalgebras  $\mathbb{S}_{01}, \mathbb{S}_{10}, \mathbb{S}_{11}$ , obtained by restricting the generators to, respectively, the sectors  $00\&01, 00\&10, 00\&11$ . The subalgebras  $\mathbb{S}_{01}, \mathbb{S}_{10}, \mathbb{S}_{11}$  are  $\mathbb{Z}_2$ -graded superdivision algebras. Since, as recalled in Section **3**, see table (15), the sectors  $01, 10, 11$  are on equal footing, the  $\mathbb{Z}_2$ -graded subalgebra projections of a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebra can be characterized by the triple

$$(\mathbb{S}_\alpha / \mathbb{S}_\beta / \mathbb{S}_\gamma), \quad (20)$$

where the order of the subalgebras is inessential.

In the following subsections we separately present the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras for, respectively, the real, complex and quaternionic series.

### 5.1 The real $\mathbb{Z}_2 \times \mathbb{Z}_2$ superdivision algebras

The four inequivalent  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras  $\mathbb{D}_{\mathbb{R};*}^{[2]}$  of the real series possess four generators. The matrix representatives of each class of equivalence can be expressed as in the table below, which gives the matrix generator of the corresponding graded sector:

	00	01	10	11
$\mathbb{D}_{\mathbb{R};1}^{[2]}$ :	$II$	$IA$	$AX$	$AY$
$\mathbb{D}_{\mathbb{R};2}^{[2]}$ :	$II$	$IA$	$YX$	$YY$
$\mathbb{D}_{\mathbb{R};3}^{[2]}$ :	$II$	$IA$	$AI$	$AA$
$\mathbb{D}_{\mathbb{R};4}^{[2]}$ :	$II$	$IY$	$YI$	$YY$

(21)

Some comments are in order.

*Comment 1* - squaring the matrices entering the 01, 10, 11 sectors gives the signs

$$\mathbb{D}_{\mathbb{R};1}^{[2]} : ---; \quad \mathbb{D}_{\mathbb{R};2}^{[2]} : +-+; \quad \mathbb{D}_{\mathbb{R};3}^{[2]} : +--; \quad \mathbb{D}_{\mathbb{R};4}^{[2]} : +++ . \quad (22)$$

*Comment 2* - the projections to the  $\mathbb{Z}_2$ -graded subalgebras, see formula (20), are given by

$$\mathbb{D}_{\mathbb{R};1}^{[2]} : (\underline{1}/\underline{1}/\underline{1}); \quad \mathbb{D}_{\mathbb{R};2}^{[2]} : (\underline{1}/\underline{2}/\underline{2}); \quad \mathbb{D}_{\mathbb{R};3}^{[2]} : (\underline{1}/\underline{1}/\underline{2}); \quad \mathbb{D}_{\mathbb{R};4}^{[2]} : (\underline{2}/\underline{2}/\underline{2}), \quad (23)$$

where, for simplicity, we denoted  $\underline{1} := \mathbb{D}_{\mathbb{R};1}^{[1]}$  and  $\underline{2} := \mathbb{D}_{\mathbb{R};2}^{[1]}$ .

*Comment 3* - the  $\mathbb{D}_{\mathbb{R};1}^{[2]}$  superdivision algebra is a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  gradation of the quaternions  $\mathbb{H}$ , the  $\mathbb{D}_{\mathbb{R};2}^{[2]}$  superdivision algebra is a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  gradation of the split-quaternions  $\tilde{\mathbb{H}}$ , the superdivision algebras  $\mathbb{D}_{\mathbb{R};3}^{[2]}$  and  $\mathbb{D}_{\mathbb{R};4}^{[2]}$  are commutative.

### 5.2 The complex $\mathbb{Z}_2 \times \mathbb{Z}_2$ superdivision algebras

The  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras  $\mathbb{D}_{\mathbb{C};*}^{[2]}$  of the complex series possess eight generators. The multiplication tables of the admissible alphabetic presentations are grouped into 5 classes of equivalence. The matrix representatives of each class of equivalence can be expressed as in the table below, which gives the matrix generators of the corresponding graded sector:

	00	01	10	11
$\mathbb{D}_{\mathbb{C};1}^{[2]}$ :	$III, IIA$	$IAX, IAY$	$AIX, AIY$	$AAI, AAA$
$\mathbb{D}_{\mathbb{C};2}^{[2]}$ :	$III, IIA$	$AIX, AIY$	$IYX, IYY$	$AYI, AYA$
$\mathbb{D}_{\mathbb{C};3}^{[2]}$ :	$III, IIA$	$YIX, YIY$	$IYX, IYY$	$YYI, YYA$
$\mathbb{D}_{\mathbb{C};4}^{[2]}$ :	$III, IIA$	$YII, YIA$	$XYI, XYA$	$AYI, AYA$
$\mathbb{D}_{\mathbb{C};5}^{[2]}$ :	$III, IIA$	$YII, YIA$	$IYI, IYA$	$YYI, YYA$

(24)

Following the construction outlined in Appendix A, by setting the  $\mathbb{Z}_2$ -graded superdivision algebras  $\mathbb{S}_{01}$  (obtained from the 00 and 01 sectors) and  $\mathbb{S}_{10}$  (obtained from the 00 and 10 sectors), the  $\mathbb{Z}_2$ -graded superdivision algebra  $\mathbb{S}_{11}$  is determined. It is convenient to denote here the three complex  $\mathbb{Z}_2$ -graded superalgebras as  $\underline{1} := \mathbb{D}_{\mathbb{C};1}^{[1]}$ ,  $\underline{2} := \mathbb{D}_{\mathbb{C};2}^{[1]}$  and  $\underline{3} := \mathbb{D}_{\mathbb{C};3}^{[1]}$ . The table below (where the first underlined number denotes  $\mathbb{S}_{01}$ , the second number  $\mathbb{S}_{10}$  and the arrow gives the  $\mathbb{S}_{11}$  output) is found:

$$\begin{aligned}
\underline{1} \times \underline{1} &\rightarrow \underline{3}, & \underline{1} \times \underline{2} &\rightarrow \underline{3}, & \underline{1} \times \underline{3} &\Rightarrow \begin{array}{c} \nearrow \underline{1} \\ \searrow \underline{2} \end{array}, \\
\underline{2} \times \underline{1} &\rightarrow \underline{3}, & \underline{2} \times \underline{2} &\rightarrow \underline{3}, & \underline{2} \times \underline{3} &\Rightarrow \begin{array}{c} \nearrow \underline{1} \\ \searrow \underline{2} \end{array}, \\
\underline{3} \times \underline{1} &\Rightarrow \begin{array}{c} \nearrow \underline{1} \\ \searrow \underline{2} \end{array}, & \underline{3} \times \underline{2} &\Rightarrow \begin{array}{c} \nearrow \underline{1} \\ \searrow \underline{2} \end{array}, & \underline{3} \times \underline{3} &\Rightarrow \underline{3}^{(*)}.
\end{aligned} \tag{25}$$

The presence of the split arrows indicates the cases where the  $\mathbb{S}_{11}$  output depends on the chosen sets of  $\mathbb{S}_{01}$ ,  $\mathbb{S}_{10}$  generators. The double arrow and the asterisk of the  $\underline{3} \times \underline{3}$  entry indicates that, depending on the chosen generators, two inequivalent  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras with the same  $\mathbb{Z}_2$ -graded projections are encountered.

With the above conventions on complex  $\mathbb{Z}_2$ -graded superdivision algebras the projections, see formula (20), to the  $\mathbb{Z}_2$ -graded subalgebras are given by

$$\mathbb{D}_{\mathbb{C};1}^{[2]} : (1/\underline{1}/\underline{3}); \quad \mathbb{D}_{\mathbb{C};2}^{[2]} : (\underline{1}/\underline{2}/\underline{3}); \quad \mathbb{D}_{\mathbb{C};3}^{[2]} : (\underline{2}/\underline{2}/\underline{3}); \quad \mathbb{D}_{\mathbb{C};4}^{[2]} : (\underline{3}/\underline{3}/\underline{3}); \quad \mathbb{D}_{\mathbb{C};5}^{[2]} : (\underline{3}/\underline{3}/\underline{3}). \tag{26}$$

*Remark:* the inequivalent  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras  $\mathbb{D}_{\mathbb{C};4}^{[2]}$ ,  $\mathbb{D}_{\mathbb{C};5}^{[2]}$  are not discriminated by their  $\mathbb{Z}_2$ -graded projections. Their difference is spotted as follows:

- i) in the  $\mathbb{D}_{\mathbb{C};4}^{[2]}$  superdivision algebra any pair of  $g, g'$  generators belonging to different 01, 10, 11 graded sectors *anticommute*;
- ii) in the  $\mathbb{D}_{\mathbb{C};5}^{[2]}$  superdivision algebra any pair of  $g, g'$  generators belonging to different 01, 10, 11 graded sectors *commute*.

### 5.3 The quaternionic $\mathbb{Z}_2 \times \mathbb{Z}_2$ superdivision algebras

The four inequivalent  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras  $\mathbb{D}_{\mathbb{H};*}^{[2]}$  of the quaternionic series possess sixteen generators. The matrix representatives of each class of equivalence are given below. In all four cases the generators of the 00 sector can be given by

$$00 : IIII, IIIA, IIAX, IIAY. \tag{27}$$

The generators of the 01, 10, 11 sectors are expressed as

	01	10	11
$\mathbb{D}_{\mathbb{H};1}^{[2]} :$	$IAII, IAIA, IAAX, IAAY$	$AXII, AXIA, AXAX, AXAY$	$AYII, AYIA, AYAX, AYAY$
$\mathbb{D}_{\mathbb{H};2}^{[2]} :$	$IAII, IAIA, IAAX, IAAY$	$AIII, AIIA, AIAX, AIAY$	$AAII, AAIA, AAAX, AAAY$
$\mathbb{D}_{\mathbb{H};3}^{[2]} :$	$IAII, IAIA, IAAX, IAAY$	$YXII, YXIA, YXAX, YXAY$	$YYII, YYIA, YYAX, YYAY$
$\mathbb{D}_{\mathbb{H};4}^{[2]} :$	$IYII, IYIA, IYAX, IYAY$	$YIII, YIIA, YIAX, YIAY$	$YAII, YAIA, YAAX, YAAY$

Let us redefine  $\underline{1}$ ,  $\underline{2}$  as the quaternionic  $\mathbb{Z}_2$ -graded superdivision algebras, so that,  $\underline{1} := \mathbb{D}_{\mathbb{H};1}^{[1]}$  and  $\underline{2} := \mathbb{D}_{\mathbb{H};2}^{[1]}$ . The quaternionic projections, see formula (20), are then given by

$$\mathbb{D}_{\mathbb{H};1}^{[2]} : (\underline{1}/\underline{1}/\underline{1}); \quad \mathbb{D}_{\mathbb{H};2}^{[2]} : (\underline{2}/\underline{2}/\underline{2}); \quad \mathbb{D}_{\mathbb{H};3}^{[2]} : (\underline{1}/\underline{1}/\underline{2}); \quad \mathbb{D}_{\mathbb{H};4}^{[2]} : (\underline{1}/\underline{2}/\underline{2}). \quad (29)$$

*Comment:* the 13 inequivalent multiplication tables of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras are straightforwardly recovered from the matrix representations of the generators, given in (21) for the real series, (24) for the complex series and (27,28) for the quaternionic series. To save space they will not be reported here.

## 6 Conclusions

In this paper we showed that the 7 inequivalent  $\mathbb{Z}_2$ -graded superdivision algebras admit an alphabetic presentation; we extended this framework to individuate 13 classes of equivalence of the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras. This result has implications both in mathematics and in physical applications. On the mathematical side we recall the investigations of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded structures (see, e.g., [34] for a state of the art account of Riemannian structures on  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded manifolds). On the physical side, the seemingly most promising application is the extension of the periodic table of topological insulators and superconductors to accommodate a  $\mathbb{Z}_2 \times \mathbb{Z}_2$  grading. In a forthcoming paper we will present the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded supercommutants, that is the algebras of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded matrices which (anti)commute with the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras. The notion of graded supercommutant is central in extending the 10 classes of fermionic Hamiltonians discussed in [5] to  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded parafermionic Hamiltonians. We recall that  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded parafermions are theoretically detectable, see [25]. The expectation is that the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded parafermionic Hamiltonians induce new features which are not encoded in the 10-fold way.

## Appendix A: alphabetic presentation of superdivision algebras

We extend here the alphabetic presentation (described in Section 2 for Clifford algebras) to the cases of  $\mathbb{Z}_2$  and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras.

Any homogeneous element  $g$  of a superdivision algebra is represented by an invertible real matrix which takes the form  $g = M \otimes N$ , where the matrix  $M$  encodes the information of the grading, either  $\mathbb{Z}_2$  or  $\mathbb{Z}_2 \times \mathbb{Z}_2$ , while the matrix  $N$  encodes the information of the real, complex or quaternionic structure.

The matrix size for  $M$  is

$$\mathbb{Z}_2\text{-grading} : (2 \times 2); \quad \mathbb{Z}_2 \times \mathbb{Z}_2\text{-grading} : (4 \times 4). \quad (A.1)$$

The matrix size for  $N$  is

$$\mathbb{R}\text{-series} : (1 \times 1); \quad \mathbb{C}\text{-series} : (2 \times 2); \quad \mathbb{H}\text{-series} : (4 \times 4). \quad (A.2)$$

Concerning the  $\mathbb{Z}_2$  grading, the even (odd) sector is denoted as  $M_0$  ( $M_1$ ); the nonvanishing elements are accommodated according to

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \in M_0, \quad \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \in M_1. \quad (A.3)$$

The tensor products of the  $\mathbb{Z}_2$ -graded matrices (A.3) produce the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded matrices  $M_{ij}$  ( $ij$  denotes the grading) according to

$$\begin{aligned} \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \otimes \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} &\mapsto \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \in M_{00}, & \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \otimes \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} &\mapsto \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \end{pmatrix} \in M_{01}, \\ \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \otimes \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} &\mapsto \begin{pmatrix} 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix} \in M_{10}, & \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} &\mapsto \begin{pmatrix} 0 & 0 & 0 & * \\ 0 & 0 & * & 0 \\ 0 & * & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix} \in M_{11}. \end{aligned} \quad (\text{A.4})$$

In the  $\mathbb{Z}_2$ -grading the  $M_0, M_1$  sectors can be spanned by the matrices denoted by the letters, with the (1) identification, given by:

$$M_0 : I, X; \quad M_1 : Y, A. \quad (\text{A.5})$$

In the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading the  $M_{00}, M_{01}, M_{10}, M_{11}$  sectors can be spanned by the matrices denoted by the 2-character words

$$\begin{aligned} M_{00} : II, IX, XI, XX; & \quad M_{01} : IA, IY, XA, XY; \\ M_{10} : AI, AX, YI, YX; & \quad M_{11} : AA, AY, YA, YY. \end{aligned} \quad (\text{A.6})$$

In Section 4 we showed that each one of the 7 inequivalent  $\mathbb{Z}_2$ -graded superdivision algebras admits an alphabetic presentation in terms of equal-length words. Without loss of generality (up to similarity transformations) the even sector  $\mathbb{D}_0^{[1]}$  can be expressed as

$$\mathbb{R}\text{-series} : I; \quad \mathbb{C}\text{-series} : II, IA; \quad \mathbb{H}\text{-series} : III, IIA, IAX, IAY. \quad (\text{A.7})$$

The odd sectors  $\mathbb{D}_1^{[1]}$  are presented in table (17) (for the  $\mathbb{R}$ -series), table (18) (for the  $\mathbb{C}$ -series), table (19) (for the  $\mathbb{H}$ -series).

The alphabetic presentation is extended to the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebras by taking into account that:

i) without loss of generality (up to similarity transformations) the even sector  $\mathbb{D}_{00}^{[2]}$  can be expressed as

$$\mathbb{R}\text{-series} : II; \quad \mathbb{C}\text{-series} : III, IIA; \quad \mathbb{H}\text{-series} : IIII, IIIA, IIAX, IIAY; \quad (\text{A.8})$$

ii) each one of the three subalgebras  $\mathbb{S}_{10}, \mathbb{S}_{01}, \mathbb{S}_{11} \subset \mathbb{D}^{[2]}$ , given by the direct sums

$$\mathbb{S}_{01} := \mathbb{D}_{00}^{[2]} \oplus \mathbb{D}_{01}^{[2]}, \quad \mathbb{S}_{10} := \mathbb{D}_{00}^{[2]} \oplus \mathbb{D}_{10}^{[2]}, \quad \mathbb{S}_{11} := \mathbb{D}_{00}^{[2]} \oplus \mathbb{D}_{11}^{[2]}, \quad (\text{A.9})$$

is isomorphic to one (of the seven)  $\mathbb{Z}_2$ -graded superdivision algebra;

iii) the alphabetic presentation can be assumed for  $\mathbb{S}_{01}$  and, since the second  $\mathbb{Z}_2$  grading is independent from the first one,  $\mathbb{S}_{10}$ . The closure under multiplication for any  $g \in \mathbb{D}_{01}^{[2]}$ ,  $g' \in \mathbb{D}_{10}^{[2]}$  implies that  $gg' \in \mathbb{D}_{11}^{[2]}$  is alphabetically presented.

As discussed in Section 5, a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded superdivision algebra can be associated with its  $\mathbb{Z}_2$ -graded superdivision algebra projections  $\mathbb{S}_{01}, \mathbb{S}_{10}, \mathbb{S}_{11}$ .

## Acknowledgments

The work was supported by CNPq (PQ grant 308095/2017-0).

## References

- [1] J. C. Baez, *The Tenfold Way*, Notices Amer. Math. Soc. **67** N. 10 (2020), 1599; arXiv:2011.14234[math.RA].
- [2] A. Kitaev, *Periodic table for topological insulators and superconductors*, AIP Conf. Proceed. **1134**, 22 (2009); arXiv:0901.2686[cond-mat.mes-hall].
- [3] M. Zirnbauer, *Riemannian symmetric superspaces and their origin in random-matrix theory*, J. Math. Phys. **37**, 4986 (1996); arXiv:math-ph/9808012.
- [4] A. Altland and M. R. Zirnbauer, *Nonstandard Symmetry Classes in Mesoscopic Normal-superconducting Hybrid Structures*, Phys. Rev. **B 55**, 1142 (1997); arXiv:cond-mat/9602137.
- [5] S. Ryu, A. P. Schnyder, A. Furusaki and W. W. Ludwig, *Topological insulators and superconductors: tenfold way and dimensional hierarchy*, New J. Phys. **12**, 065010 (2010); arXiv:0912.2157[cond-mat.mes-hall].
- [6] D. S. Freed and G. W. Moore, *Twisted Equivariant Matter*, Ann. Henri Poincaré **14**, 1927 (2013); arXiv:1208.5055[hep-th].
- [7] F. J. Dyson, *The threefold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics*, J. Math. Phys. **3**, 1199 (1962).
- [8] C. T. C. Wall, *Graded Brauer groups*, J. Reine angew. Math. **213**, 187 (1964).
- [9] V. Rittenberg and D. Wyler, *Generalized Superalgebras*, Nucl. Phys. **B 139**, 189 (1978).
- [10] V. Rittenberg and D. Wyler, *Sequences of  $\mathbb{Z}_2 \otimes \mathbb{Z}_2$  graded Lie algebras and superalgebras*, J. Math. Phys. **19**, 2193 (1978).
- [11] M. Scheunert, *Generalized Lie algebras*, J. Math. Phys. **20**, 712 (1979).
- [12] N. Aizawa, Z. Kuznetsova, H. Tanaka and F. Toppan,  *$\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie symmetries of the Lévy-Leblond equations*, Prog. Theor. Exp. Phys. **2016**, 123A01 (2016); arXiv:1609.08224[math-ph].
- [13] N. Aizawa, Z. Kuznetsova, H. Tanaka and F. Toppan, *Generalized supersymmetry and Lévy-Leblond equation*, in S. Duarte *et al* (eds), Physical and Mathematical Aspects of Symmetries, Springer, Cham, p. 79 (2017); arXiv:1609.08760[math-ph].
- [14] A. J. Bruce and S. Duplij, *Double-graded supersymmetric quantum mechanics*, J. Math. Phys. **61**, 063503 (2020); arXiv:1904.06975 [math-ph].
- [15] N. Aizawa, Z. Kuznetsova and F. Toppan,  *$\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded mechanics: the classical theory*, Eur. J. Phys. **C 80**, 668 (2020); arXiv:2003.06470[hep-th].
- [16] N. Aizawa, Z. Kuznetsova and F. Toppan,  *$\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded mechanics: the quantization*, Nucl. Phys. **B 967**, 115426 (2021); arXiv:2005.10759[hep-th].
- [17] A. J. Bruce,  *$\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded supersymmetry: 2-d sigma models*, J. Phys. A: Math. Theor. **53**, 455201 (2020); arXiv:2006.08169[math-ph].
- [18] Z. Kuznetsova and F. Toppan, *Classification of minimal  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie (super)algebras and some applications*, J. Math. Phys. **62**, 063512 (2021); arXiv:2103.04385[math-ph].

- [19] A. J. Bruce, *Is the  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded sine-Gordon equation integrable?*, Nucl.Phys. **B 971**, 115514 (2021); arXiv:2106.6372[math-ph].
- [20] N. Aizawa, K. Amakawa and S. Doi,  *$\mathcal{N}$ -Extension of double-graded supersymmetric and superconformal quantum mechanics*, J. Phys. A: Math. Theor. **53**, 065205 (2020); arXiv:1905.06548[math-ph].
- [21] A. J. Bruce and S. Duplij, *Double-graded quantum superplane*, Rep. on Math. Phys. v. **86** n. 3, 383 (2020); arXiv:1910.12950[math.QA].
- [22] N. Aizawa and S. Doi, *Comments of  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -supersymmetry in superfield formalism*, arXiv:2109.14227[math-ph].
- [23] N. Aizawa, K. Amakawa and S. Doi,  *$\mathbb{Z}_2^n$ -graded extensions of supersymmetric quantum mechanics*, J. Math. Phys. **61**, 052105 (2020); arXiv:1912.11195[math-ph].
- [24] C. Quesne, *Minimal bosonization of double-graded quantum mechanics*, Mod. Phys. Lett. **A 36**, 2150238 (2021); arXiv:2108.06243[math-ph].
- [25] F. Toppan,  *$\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded parastatistics in multiparticle quantum Hamiltonians*, J. Phys. A: Math. Theor. **54**, 115203 (2021); arXiv:2008.11554[hep-th].
- [26] F. Toppan, *Inequivalent quantizations from gradings and  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded parabosons*, J. Phys. A: Math. Theor. **54**, 355202 (2021); arXiv:2104.09692[hep-th].
- [27] V. N. Tolstoy, *Once more on parastatistics*, Phys. Part. Nucl. Lett. **11**, 933 (2014); arXiv:1610.01628[math-ph].
- [28] N. I. Stoilova and J. Van der Jeugt, *The  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -graded Lie superalgebra  $ps(2m + 1|2n)$  and new parastatistics representations*, J. Phys. A: Math. Theor. **51**, 135201 (2018); arXiv:1711.02136[math-ph].
- [29] F. Toppan and P. W. Verbeek, *On Alphabetic Presentations of Clifford Algebras and Their Possible Applications*, J. Math. Phys. **50**, 123523 (2009); arXiv:0903.0940[math-ph].
- [30] S. Okubo, *Real representations of finite Clifford algebras. I. Classification*, J. Math. Phys. **32**, 1657 (1991).
- [31] M. F. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology **3** Supp. 1, 3 (1964).
- [32] H. L. Carrion, M. Rojas and F. Toppan, *Quaternionic and Octonionic Spinors. A Classification*, JHEP **0304**, 040 (2003); arXiv:hep-th/0302113.
- [33] K. McCrimmon, *A Taste of Jordan Algebras*, Universitext Springer, New York (2004).
- [34] A. J. Bruce and J. Grabowski, *Riemannian structures on  $\mathbb{Z}_2^n$ -manifolds*, Mathematics **8**, 1469 (2020); arXiv:2007.07666.